Sample final solutions

Q1 a) $\int_0^\infty x e^{-3x^2} dx = \lim_{R\to\infty} \int_0^R x e^{-3x^2} dx$

$\int x e^{-3x^2} dx$ by $u = -3x^2$, $\frac{du}{dx} = -6x$ $\int x e^u \frac{dx}{du} du = \int x e^u \frac{1}{-6x} du$

$= -\frac{1}{6}\int e^u du = -\frac{1}{6} e^u + c = -\frac{1}{6} e^{-3x^2} + c$

$\lim_{R\to\infty}\left[-\frac{1}{6}e^{-3x^2}\right]_0^R = \lim_{R\to\infty} -\frac{1}{6}e^{-3R^2} + \frac{1}{6} = \frac{1}{6}$

b) $\int \underbrace{x}_{u}\underbrace{e^{-3x}}_{v} dx$ $\int uv' dx = uv - \int u'v dx$ $u = x$ $u' = 1$ $v' = e^{-3x}$ $v = -\frac{1}{3}e^{-3x}$

$\int x e^{-3x} dx = -\frac{1}{3} x e^{-3x} + \int \frac{1}{3} e^{-3x} dx = -\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} + c$

$\int_0^\infty x e^{-3x} dx = \lim_{R\to\infty} \int_0^R x e^{-3x} dx = \lim_{R\to\infty}\left[-\frac{1}{3}x e^{-3x} - \frac{1}{9}e^{-3x}\right]_0^R$

$= \lim_{R\to\infty} -\frac{1}{3} R e^{-3R} - \frac{1}{9}e^{-3R} + \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$

c) $\int \sin^3 x \cos^2 x \, dx$ $u = \cos x$, $u' = -\sin x$ $\int \sin x (1-u^2) u^2 \frac{dx}{du} du$

$\sin x(\sin^2 x) = \sin x(1-\cos^2 x)$

$= \int \sin x (u^2 - u^4)\frac{1}{-\sin x} du = \int u^4 - u^2 du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$

$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$

d) $\int \sin 5x \cos 4x \, dx$

$$\left.\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A \sin B\\ \sin(A-B) &= \sin A\cos B - \cos A\sin B\end{aligned}\right\} \begin{aligned}&\sin(A+B)\\ &+\sin(A-B)\end{aligned} = 2\sin A\cos B.$$

$= \frac{1}{2}\int \sin 9x + \sin x \, dx = -\frac{1}{18}\cos 9x - \frac{1}{2}\cos x + c$

Q2 $f(x) = e^{2x}\ln(x)$

$f'(x) = 2e^{2x}\ln(x) + e^{2x}\cdot\frac{1}{x}$

$f''(x) = 4e^{2x}\ln(x) + 2e^{2x}\frac{1}{x} + 2e^{2x}\cdot\frac{1}{x} + e^{2x}\cdot -\frac{1}{x^2}$

$f^{(3)}(x) = 8e^{2x}\ln(x) + 8e^{2x}\frac{1}{x} + 4e^{2x}\cdot -\frac{1}{x^2} + 2e^{2x}\cdot -\frac{1}{x^2} + e^{2x}\cdot\frac{2}{x^3}$

$f(1) = 0$

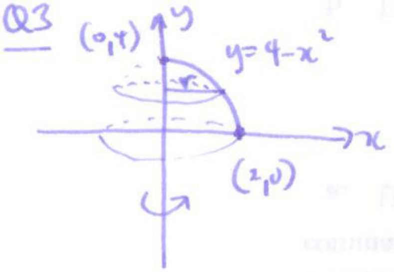
$f'(1) = e^2$

$f''(1) = 3e^2$

$f^{(3)}(1) = 4e^2$

$$T_3 = 0 + e^2(x-1) + \frac{3e^2}{2!}(x-1)^2 + \frac{4e^2}{3!}(x-1)^3$$

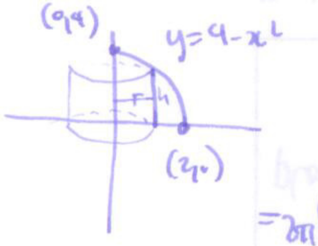
Q3



a) $\int \pi r^2\,dy = \pi\int_0^4 x^2\,dy = \pi\int_0^4 4-y\,dy$

$= \pi\left[4y - \frac{1}{2}y^2\right]_0^4 = \pi\left(16 - \frac{1}{2}16\right) = 8\pi.$

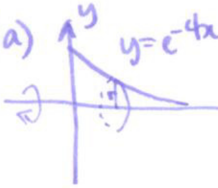
b)



$\int 2\pi rh\,dx = 2\pi\int_0^2 xy\,dx = 2\pi\int_0^2 x(4-x^2)dx$

$= 2\pi\int_0^2 4x - x^3\,dx = 2\pi\left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 2\pi(8-4) = 8\pi.$

Q4 a)



$\int_0^\infty \pi r^2\,dx = \lim_{R\to\infty}\pi\int_0^R e^{-8x}\,dx = \lim_{R\to\infty}\pi\left[-\frac{1}{8}e^{-8x}\right]_0^R$

$= \lim_{R\to\infty}\pi\left(-\frac{1}{8}e^{-8R} + \frac{1}{8}\right) = \frac{\pi}{8}$

b) surface area $= \int_0^\infty 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)}\,dx = \int_0^\infty 2\pi e^{-4x}\sqrt{1+16e^{-8x}}\,dx.$

Q5 $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad xe^{-x^2} = x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \cdots$ $\quad a_n = \frac{(-1)^n x^{2n+1}}{n!}$

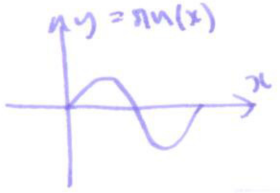
$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$

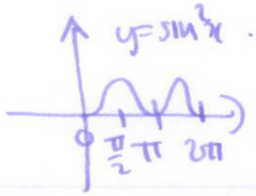
ratio test

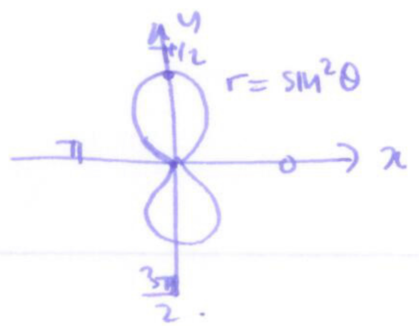
$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!}\cdot\frac{n!}{x^{2n+1}}\right| = \lim_{n\to\infty}|x^2|\frac{1}{n+1} = 0 < 1$

so radius of convergence is $R = \infty$.

Q6







$$\text{area} = \int_0^{2\pi} \frac{1}{2}\sin^4\theta \, d\theta$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\int \sin^4\theta\,d\theta = \int \underbrace{\sin\theta}_{v'} \cdot \underbrace{\sin^3\theta}_{u}\,d\theta = \quad -\cos\theta\sin^3\theta + \int \cos\theta\, 3\sin^2\theta \cdot \cos\theta\,d\theta$$

$$\int \sin^4\theta\,d\theta = -\cos\theta\sin^3\theta + 3\int \sin^2\theta - \sin^4\theta\,d\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$

$$4\int \sin^4\theta\,d\theta = -\cos\theta\sin^3\theta + 3\int \sin^2\theta\,d\theta = -\cos\theta\sin^3\theta + \frac{3}{2}\int 1 - \cos 2\theta\,d\theta$$

$$\int \sin^4\theta\,d\theta = -\frac{1}{4}\cos\theta\sin^3\theta + \frac{3}{8}\theta - \frac{1}{8}\sin 2\theta + c$$

$$\text{area} = \frac{1}{2}\left[-\frac{1}{4}\cos\theta\sin^3\theta + \frac{3}{8}\theta - \frac{1}{8}\sin 2\theta\right]_0^{2\pi} = \frac{3}{8}\pi.$$

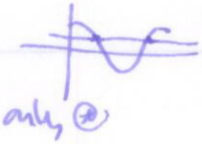
tangents: (r,θ) $\quad (\sin^2\theta, \theta)$ with r under $\sin^2\theta$ and θ under θ —

$(\underset{r}{\sin^2\theta}, \underset{\theta}{\theta})$ $\quad x = r\cos\theta = \sin^2\theta\cos\theta \quad \frac{dx}{d\theta} = 2\sin\theta\cos^2\theta 4 - \sin^3\theta$

$y = r\sin\theta = \sin^3\theta \quad \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta.$

$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$ vertical when $\frac{dx}{d\theta} = 0$ $\text{and } \frac{dy}{d\theta} \neq 0$ $\quad \sin\theta(2\cos^2\theta - \sin^2\theta) = 0 \quad \sin\theta(3\cos^2\theta - 1) = 0$

$\sin\theta = 0, \quad 0, \pi, 2\pi \quad \cos\theta = \frac{1}{\sqrt[3]{3}}$



$\theta = \cos^{-1}\left(\frac{1}{\sqrt[3]{3}}\right), 2\pi - \cos^{-1}\left(\frac{1}{\sqrt[3]{3}}\right)$ ⊛.

check $\frac{dy}{dx} = \frac{3\sin\theta\cos\theta}{2\cos^2\theta - \sin^2\theta} = 0$ for $0, \pi, 2\pi$, so only ⊛

Q7 a) $\begin{array}{r} x \\ x\overline{)x^2+3} \end{array}$ $\quad \int \frac{x^2+3}{x}\,dx = \int x + \frac{3}{x}\,dx = \frac{1}{2}x^2 + 3\ln|x| + c$

b) $\begin{array}{r} x-1 \\ x+1\overline{)x^2+2} \\ \underline{x^2+x} \\ -x+2 \\ \underline{-x-1} \\ 3. \end{array}$ $\quad \int \frac{x^2+2}{x+1}\,dx = \int x - 1 + \frac{3}{x+1}\,dx = \frac{1}{2}x^2 - x + 3\ln|x+1| + c$

c) $\int \frac{x}{3x^2+1}\,dx \quad u = 3x^2+1 \quad \frac{du}{dx} = 6x \quad \int \frac{x}{u}\cdot\frac{dx}{du}\,du = \int \frac{x}{u}\cdot\frac{1}{6x}\,dx = \int \frac{1}{6}\,\frac{1}{u}\,du$

$= \frac{1}{6}\ln|u| + c = \frac{1}{6}\ln|3x^2+1| + c$

d) $\int \frac{1}{1+4x^2}\,dx$ $x=\frac{1}{2}\tan\theta$ $\frac{dx}{d\theta}=\frac{1}{2}\sec^2\theta$ $\cos^2\theta+\sin^2\theta=1$

$1+\tan^2\theta=\sec^2\theta$

$$\int \frac{1}{1+\tan^2\theta}\frac{dx}{d\theta}d\theta = \int \frac{1}{\sec^2\theta}\frac{1}{2}\sec^2\theta\,d\theta = \int \frac{1}{2}d\theta = \frac{1}{2}\theta + c = \frac{1}{2}\tan^{-1}(2x)+c$$

Q8 a) converges, geometric series with $r = -\frac{\sqrt{2}}{e} < 1$.

b) $\sum \frac{1}{2+n^2}$ comparison test $\frac{1}{2+n^2} < \frac{1}{n^2}$ $\sum \frac{1}{n^2}$ converges by p-series $p>1$.

$\Rightarrow \sum \frac{1}{2+n^2}$ converges.

c) $\sum \frac{(-1)^n}{2+n^2}$ alternating series test $a_n = \frac{1}{2+n^2}$ positive, decreasing $a_n \to 0$ as $n\to\infty$.

$\Rightarrow$ converges.

d) $\sum_{n=0}^{\infty} \frac{10^n}{n!}$ $a_n = \overbrace{\frac{10 \ldots\ 10}{1\cdot 2 \ldots\ 10}}^{A} \cdot \frac{10}{11}\cdot\frac{10}{12}\cdot \ldots\ \frac{10}{(n-1)}\cdot\frac{10}{n} < \frac{100A}{(n-1)n} < \frac{10^{12}}{(n-1)n}$.

comparison test: $\sum \frac{10^{12}}{(n-1)n}$ converges: use limit ratio test with $\frac{1}{n^2} \in$ converges p-series. $p>1$.

$$\lim_{n\to\infty} \frac{10^{12}}{(n-1)n}\cdot n^2 = \lim_{n\to\infty} 10^{12}\frac{n^2}{n^2-n} = 10^{12}\lim_{n\to\infty}\frac{1}{1-1/n} = 10^{12}.$$

so $\sum \frac{10^{12}}{(n-1)n}$ converges $\Rightarrow \sum_{n=0}^{\infty}\frac{10^n}{n!}$ converges.

Q9 a) $a_n = 2-\frac{1}{n+1}$ $\lim_{n\to\infty} 2-\frac{1}{n+1} = 2$ sequence converges.

b) $\sum_{n=0}^{\infty} 2-\frac{1}{n+1}$ does not converge as $a_n \not\to 0$.

Q10 $y=2x^2$ $(0,0)$ $(1,2)$ $x(t)=t$ $\frac{dx}{dt}=1$

$y(t)=2t^2$ $\frac{dy}{dt}=4t$

a) arc length $= \int_0^1 \sqrt{1+16t^2}\,dt$ try $t=\frac{1}{4}\tan\theta$ $\frac{dt}{d\theta}=\frac{1}{4}\sec^2\theta$.

$$= \int_0^{\tan^{-1}(4)} \sqrt{1+\tan^2\theta}\ \frac{1}{4}\sec^2\theta\,d\theta = \int_0^{\tan^{-1}(4)} \frac{1}{4}\sec^3\theta\,d\theta$$

⑤

$$\int \sec^3\theta \, d\theta = \int \underbrace{\sec\theta}_{u} \, \underbrace{\sec^2\theta}_{v'} \, d\theta = \sec\theta \tan\theta - \int \sec\theta \underbrace{\tan\theta \tan\theta}_{\cancel{1-\sec^2\theta}} \, d\theta$$

$4+\sec^2\theta-1$

$\sin^2\theta+\cos^2\theta=1$

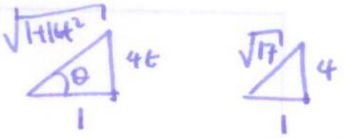
$\tan^2\theta+1=\sec$

$\int uv'dx = uv - \int u'v dx$

$u=\sec\theta \quad u'=\sec\theta\tan\theta$

$v'=\sec^2\theta \quad v=\tan\theta$

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \sec\theta + \sec^3\theta \, d\theta$$



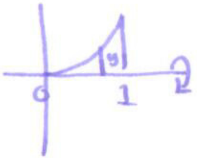
$$2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + c$$

$\tan\theta = 4 \quad \sec\theta = \frac{1}{\sqrt{17}}.$

$$\frac{1}{4}\int \sec^3\theta \, d\theta = \frac{1}{8}\sec\theta\tan\theta + \frac{1}{8}\ln|\sec\theta+\tan\theta| + c$$

$$\int_0^{\tan^{-1}(4)} \frac{1}{4}\sec^3\theta \, d\theta = \frac{1}{8}\frac{1}{\sqrt{17}}\cdot 4 + \frac{1}{8}\ln\left|\frac{1}{\sqrt{17}}+4\right| - 0 - \frac{1}{8}\ln\left|\frac{1}{\sqrt{17}}\right|.$$

b) surface area $= 2\pi \int_0^1 2t^2\sqrt{1+16t^2}\, dt. \qquad t=\frac{1}{4}\tan\theta \cdot \frac{dt}{d\theta}=\frac{1}{4}\sec^2\theta.$



$$= 2\pi\int \frac{1}{8}\tan^2\theta \underbrace{\sqrt{1+\tan^2\theta}}_{\sec^2\theta} \cdot \frac{1}{4}\sec^2\theta \, d\theta$$

$$= \frac{\pi}{16}\int \underset{\sec^2\theta-1}{\tan^2\theta} \sec^3\theta \, d\theta = \frac{\pi}{16}\int \sec^5\theta - \sec^3\theta \, d\theta \quad \circledast.$$

$$\int \sec^5\theta \, d\theta = \int \sec^2\theta \cdot \sec^3\theta \, d\theta = \tan\theta\sec^3\theta - \int \underbrace{\tan\theta \cdot 3\sec^2\theta \sec\theta \tan\theta}_{3\underset{\sec^2\theta-1}{\tan^2\theta}\sec^3\theta} \, d\theta$$

$$\int \sec^5\theta \, d\theta = \tan\theta\sec^3\theta - \int 3\sec^5\theta - 3\sec^3\theta \, d\theta$$

$$4\int \sec^5\theta \, d\theta = \tan\theta\sec^3\theta + 3\int\sec^3\theta \, d\theta$$

$$\circledast = \frac{\pi}{16}\left(\tan\theta\sec^3\theta + 2\int\sec^3\theta \, d\theta\right) = \frac{\pi}{16}\left(\tan\theta\sec^3\theta + \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)$$

$$\int_0^1 \frac{\pi}{16}\tan^2\theta\sec^3\theta \, d\theta = \frac{\pi}{16}\left(4\frac{1}{(\sqrt{17})^3} + \frac{1}{\sqrt{17}}4 + \ln\left|\frac{1}{\sqrt{17}}+4\right| - \ln\left|\frac{1}{\sqrt{17}}\right|\right).$$