(Gromov) norm on homology

$H_k(X;\mathbb{R})$ is an ($\mathbb{R}$-)vector space (use singular homology)

$\alpha = [c]$ where $c = \sum_{i=1}^{n} r_i \sigma_i$, set $\|c\| = \sum_{i=1}^{n} |r_i|$. ($L^1$-norm).

set $\|\alpha\| = \inf_{[c]=\alpha} \|c\|$

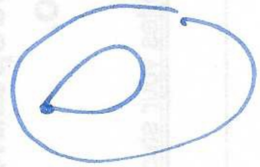
$\mathbb{Q}$ dense in $\mathbb{R}$, so suffices to consider $\mathbb{Q}$ coeffs.
$q \in \mathbb{Q}$ ~~can be approx by~~ $= \frac{a}{b}$.
so suffices to consider $n\alpha \in H_k(X;\mathbb{R})$.

claim $\|\cdot\|$ is a semi-norm on $H_k(X;\mathbb{R})$.

- subadditive $\|\alpha+\beta\| \leq \|\alpha\| + \|\beta\|$.
- linear $\|k\alpha\| = k\|\alpha\|$.

$f: X \to Y$ then $\alpha \in H_k(X;\mathbb{R})$ $\|f_*(\alpha)\| \leq \|\alpha\|$

Example $H_1(X;\mathbb{R}) \ni \alpha$ then $\|\alpha\| = 0$.



X.

α represented by a loop. $\sigma: I \longrightarrow X$

$\sigma^2: I \longrightarrow X$ represents 2α.

so $\frac{1}{2}\sigma^2$ represents α so $\|\alpha\| \leq \frac{1}{2}$.

$\sigma^n: I \longrightarrow X$ represent $n\alpha$ so $\|\alpha\| = 0$.

Example $H_2(T^2;\mathbb{R})$

fundamental class: $\|[T]\| \leq 2$.

$\|4[T]\| \leq 1 \Rightarrow \|[T]\| \leq \frac{1}{4}$.

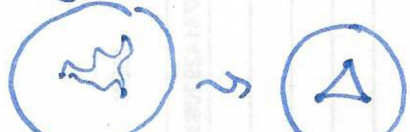
$\Rightarrow \|[T]\| = 0$.

Example $H_2(S_2;\mathbb{R})$

$\leftarrow$ universal cover $\mathbb{H}^2$.

straightening map: $\sigma: \Delta \to S_2$

$\tilde{\sigma} \to \mathbb{H}^2$
$\downarrow$
$\sigma: \Delta \to S_2$



$\leftarrow$ area of triangle $\leq \pi$.

Fact: straightening map gives a chain complex homotopy

Fact: Gauss-Bonnet: $\text{area}(S) = -2\pi\chi(S)$ $\Rightarrow$ $\|[S_2]\| \geq 4$.

Fact: $\|[S_g]\| = -2\chi(S)$

Stable commutator length and bounded cohomology (after Calegari)

$F_2 \to \mathbb{Z}^2$

G = group $\quad G' = [G,G]$ commutator subgroup (ker (abelianization))
- group generated by all commutators. $[a,b] = aba^{-1}b^{-1}$

Def$^{\underline{n}}$ $cl(g) = \min\{ n \mid g$ is a product of n commutators$\}$

- subadditive : $cl(gh) \leq cl(g) + cl(h)$

Def$^{\underline{n}}$ $scl(g) = \lim_{n \to \infty} \frac{1}{n} cl(g^n)$

- $scl(g^n) = n\, scl(g)$, $scl(hgh^{-1}) = scl(g)$
- $\phi : G \to H$ homomorphism, then $scl(\phi(g)) \leq scl(g)$.

Remarks :
- $cl(g)$ word length on $[G,G]$ wrt generating set of all commutators.
- $scl(g)$ translation length of g acting on G'.

Example $F_2 = \langle a,b \mid \rangle \quad g = [a,b] = aba^{-1}b^{-1} \quad cl(g) = 1$

$cl(g^2) = 2 \quad cl(g^3)? \quad [a,b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, b^2]$

$cl(g^3) = 2 \quad (\Rightarrow scl(g) \leq \frac{2}{3}) \quad$ **Fact** $scl(g) = \frac{1}{2}$.

topological version

group $G \leftrightarrow$ space X, $\pi_1 X = G$

$g \in G \leftrightarrow$ loop γ in G

conjugacy class of $g \leftrightarrow$ free homotopy class of γ

$g \in G' \leftrightarrow [\gamma] = 0 \in H_1(X) \leftrightarrow \gamma$ bounds a surface $S \hookrightarrow X$ s.t. $\partial S \to \gamma$ is deg 1

$cl(g) \leftrightarrow \inf\{\text{genus}(S) \mid S \hookrightarrow X \text{ s.t. } \partial S \to \gamma \text{ is deg } 1\}$

$scl(g) \leftrightarrow \inf\{\frac{1}{n}\text{genus}(S) \mid S \hookrightarrow X \text{ s.t. } \partial S \to \gamma \text{ is deg } n\}$

better **Def$^{\underline{n}}$** $scl(g) = \inf\{-\frac{1}{2n}\chi(S) \mid \partial S \to \gamma \text{ deg } n\}$

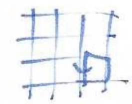
(so $scl([a,b]) \leq \frac{1}{2}$ in F_2).

~~Fact~~ Remarks: scl is dual to quasimorphisms.

Defn $\phi : G \to \mathbb{R}$ quasimorphism if $|\phi(gh) - \phi(g) - \phi(h)| \leq D(\phi)$ $\forall g, h \in G$. $D(\phi)$ is called the defect of ϕ.

$\bar{Q}$ homogeneous $\subset Q$. $\phi(a^n) = \phi(a)^n$.

Fact [Bavard duality] $$scl(g) = \frac{1}{2} \sup_{\phi \in Q/H^1} \frac{|\phi(g)|}{D(\phi)}$$



Example $\phi : F_2 \to \mathbb{Z} \subset \mathbb{R}$ "winding number" $= \left(\#\text{left turns} - \#\text{right turns} \right) / 4$.

claim $D(\phi) = 1$ so $scl([a,b]) \geq \frac{1}{2} \phi([a,b]) = \frac{1}{2} \Rightarrow scl([a,b]) = \frac{1}{2}$.

Bounded cohomology G group $\rightsquigarrow$ simplicial $K(G,1)$ ~~with simplices~~ $1 \triangle$

$EG = [g_0, g_1, \dots g_n]$ simplices. bounding map ef. (contractible) $[g_0, \dots, g_n] < [1, g_0, \dots g_n]$

$G \curvearrowright EG$ as a covering space action $K(G,1) = BG = EG/G$.

normalization: $[h_0, h_1, \dots, h_n] \leftrightarrow h_0 [1, g_1, g_1 g_2, \dots, g_1 \cdots g_n]$. where $g_i = h_{i-1}^{-1} h_i$

bar notation: $[g_1 | g_2 | \dots | g_n]$ $\leftarrow$ gives bar complex $C_*(G)$

Exercise $\partial([h_0, \dots h_n]) = \sum_{i=0}^{n} (-1) [h_0, \dots \hat{h_i}, \dots, h_n]$ *

$$\partial [g_1 | \dots | g_n] = [g_2, \dots g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1, \dots g_i g_{i+1}, \dots, g_n] + (-1)^n (g_1 \cdots g_{n-1})$$

dual cochain complex $C^*(G; R) = \mathrm{Hom}(C_*(G), R)$

$\uparrow$

$H^*(G;R)$ group cohomology w/ coeffs in $R \subset \mathbb{R}$

$\alpha \in C^n(G)$ is <u>bounded</u> if $\sup |\alpha(g_1 | \dots | g_n)| < \infty$

norm $\|\alpha\|_\infty = \sup |\alpha(g_1 | \dots | g_n)|$

set of all bounded cochains is a subcomplex of $C^*(G)$

$C_b^*(G) < C^*(G)$, so $C_b^n(G)$ is a Banach space

note: for $[\alpha] \in H_b^*(G)$ define $\|[\alpha]\|_\infty = \inf_{\sigma \in [\alpha]} \|\sigma\|_\infty$ $\leftarrow$ function not nec. norm as $\|[\alpha]\|_\infty = 0$ $\alpha \neq 0$ possible.

note: most cochains not bounded. example volume in $\mathbb{H}^n$.

note: $H^1(G;\mathbb{R}) = \mathrm{Hom}(G,\mathbb{R})$: $C^2(G) \xleftarrow{\delta} C^1(G) \xleftarrow{\delta} C^0(G)$
$\delta\phi$. ϕ.

$\ker(\delta)/\mathrm{Im}(\delta)$ = constant functions on G.

$$\delta\phi([g_1|g_2]) = \phi(\partial[g_1|g_2]) = \phi([g_2]-[g_1g_2]+[g_1]) = \phi([g_1])+\phi([g_2])-\phi([g_1g_2])$$

$\delta\phi = 0 \Rightarrow \phi([g_1g_2]) = \phi([g_1])+\phi([g_2])$. $\Rightarrow \phi \in \mathrm{Hom}(G,\mathbb{R})$.

note: $\Rightarrow H^1_b(G;\mathbb{R}) = 0$ as any bounded $\mathrm{Hom}(G,\mathbb{R})$ is zero!

Thm $0 \to H^1(G;\mathbb{R}) \to \overline{Q}(G) \to H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ is exact.

Proof short exact sequence of chain complexes $0 \to C^*_b \to C^* \to C^*/C_b \to 0$ gives a long exact sequence of cohomology groups:

$$\begin{array}{l} \to H^2_b(G;\mathbb{R}) \to H^2(G;\mathbb{R}) \to H^2(C^*/C_b) \\ H^1_b(G;\mathbb{R}) \to H^1(G;\mathbb{R}) \to H^1(C^*/C_b) \\ \overset{\shortparallel}{0} \end{array}$$

$$C^2/C^2_b \xleftarrow{\delta} C^1/C^1_b \xleftarrow{\delta} C^0/C^0_b$$
$\delta\phi$ ϕ

C^1 = function $G \to \mathbb{R}$
C^1_b = bounded functions $G \to \mathbb{R}$

$\delta\phi = \phi(\partial\,\cdot)$ $\quad \delta\phi([g_1|g_2]) = \underbrace{\phi([g_1]) - \phi([g_1g_2]) + \phi([g_2])}_{\in C^2_b} = 0$ in C^2/C^2_b.

i.e. $\exists\, D(\phi)$ s.t. $|\phi(g_1g_2) - \phi(g_1) - \phi(g_2)| \leq D(\phi)$ i.e. $\phi \in Q$.

$H^1(C^*/C^*_b) = Q/C^1_b = \overline{Q}$ as if $\psi = \phi + \beta$ then $\overline{\psi} = \overline{\phi}$

where $\overline{\phi}(g) = \lim_{n\to\infty} \frac{1}{n}\phi(g^n)$ is the homogenization. $\square$.