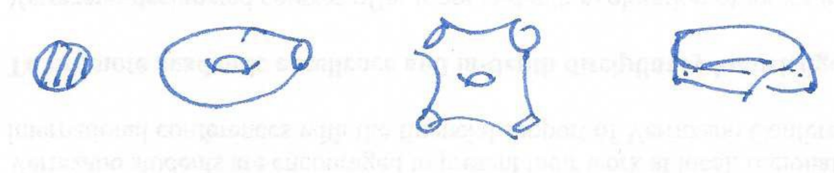
Other forms ~~for~~ of duality

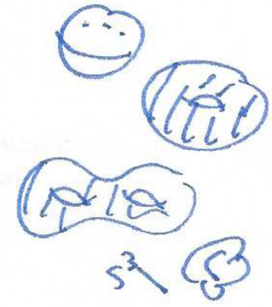
Defn M is a manifold with boundary if M is Hausdorff, 2nd countable topological space and every $x \in M$ has an open nbd homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times [0,\infty)$

these points are called the boundary ∂M

Examples $n=2$



$n=3$



Remark if $\partial M \neq \emptyset$ then $H_n(M) = 0$

as $M \simeq M \setminus \partial M$ not compact. (or deformation retract to $(n-1)$-dim complex)

homotopy equivalent

Thm (Poincaré–Lefschetz duality). M compact n-manifold, possibly w/ boundary

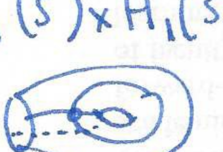
(A) $H^k(M) \cong H_{n-k}(M, \partial M)$

(B) $H^k(M, \partial M) \cong H_{n-k}(M)$

(C) $H^k(M, N_1) \cong H_{n-k}(M, N_2)$ where $\partial M = N_1 \cup N_2$.

Geometric picture (cap product on homology)

$H_1(S) \times H_1(S, \partial S) \to \mathbb{Z}$ but not $H_1(S, \partial S) \times H_1(S, \partial S) \to \mathbb{Z}$



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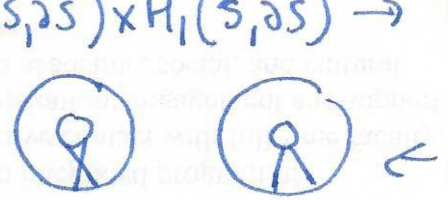


intersection # ← not well defined

Thm (Alexander duality) If $K \subseteq S^n$ is compact, locally contractible and not $\emptyset$ or S^n, then $\tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$ for all i.

Corollary $\tilde{H}_*(S^n \setminus K)$ does not depend on the embedding of K in S^n.

$S^1 \hookrightarrow S^3$

or wild knot! ($\pi_1(S^3 \setminus K)$ not finitely generated)

$\tilde{H}^i_*(S^3 \setminus S^1) = \mathbb{Z}$ if $i=1$, 0 else.

note: works for wild knots w/ $\pi_1(S^3 \setminus K)$ not finitely generated.

Proof (sketch)

$$H_i(S^n \setminus K) \cong H_c^{n-i}(S^n \setminus K) \quad \text{(Poincaré duality)}$$

$$= \varinjlim_{\substack{U \supseteq K \\ \text{open}}} H^{n-i}(S^n \setminus K, U \setminus K)$$

$$\cong \varinjlim H^{n-i}(S^n \setminus K / U \setminus K) \quad \text{(excision)}$$

$$= \varinjlim H^{n-i}(S^n, U) \quad \text{(excision)}.$$

$$= \varinjlim H^{n-i-1}(U) \quad \text{(long exact sequence of a pair)}$$

$$= \tilde{H}^{n-i-1}(K) \leftarrow \text{easy to see if } U \text{ deformation retracts to } K \text{, in general only have retracts, see Hatcher A.7.} \quad \square$$

Application: $\partial M \xhookrightarrow{i} M$ $\quad H_1(\partial M) \to H_1(M)$ has kernel for M^3 compact, $\partial M \neq \emptyset$.

$$\cdots \to H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \xrightarrow{i_*} H_1(M) \xrightarrow{\delta} H_1(M, \partial M) \to \cdots$$

$$\cong \qquad \cong \qquad \cong \qquad \cong \qquad \cong$$

$$\cdots \to H^1(M, \partial M) \to H^1(M) \xrightarrow{i^*} H^1(\partial M) \to H^2(M, \partial M) \to H^2(M) \to \cdots$$

assume: $H^k(M)$ etc torsion free (otherwise use $\mathbb{Z}_p$ or $\mathbb{Q}$ coeffs).

recall. $\quad \cdots \to \mathbb{Z}^a \xrightarrow{\alpha} \mathbb{Z}^b \xrightarrow{\beta} \mathbb{Z}^c \to \cdots$ exact $\quad \dim(\alpha) + \dim(\beta) = b$.

so $\dim(i_*) + \dim(i_*) \overset{\dim}{=} H_1(\partial M)$. so $\dim(\ker(i_*)) = \frac{1}{2} \dim H_1(M)$.

"half lives / half dies" same analogue in odd dimensions...

§4 Homotopy groups

s_0 basepoint in S^n

X topological space with basepoint x_0

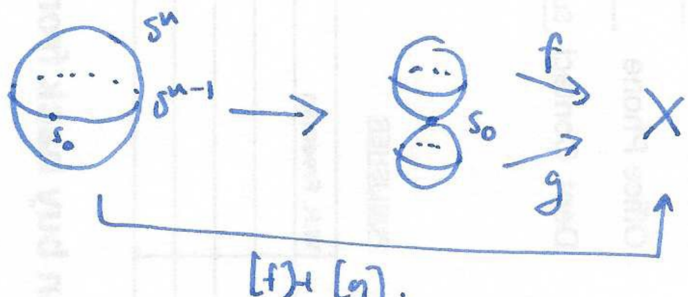
$\pi_n(X, x_0)$ = homotopy classes of maps $(S^n, s_0) \to (X, x_0)$.

$\pi_1(X, x_0)$ = fundamental group

$\pi_0(X, x_0) = \{ S^0 = \overset{s_0}{\cdot} \; \overset{s_1}{\cdot} \xrightarrow{f} X$ 

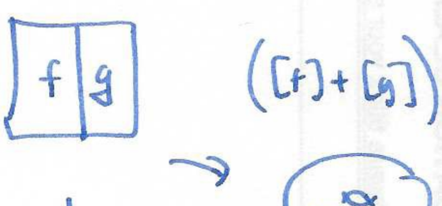
= set of path components of X.

Composition for $n \geqslant 2$:

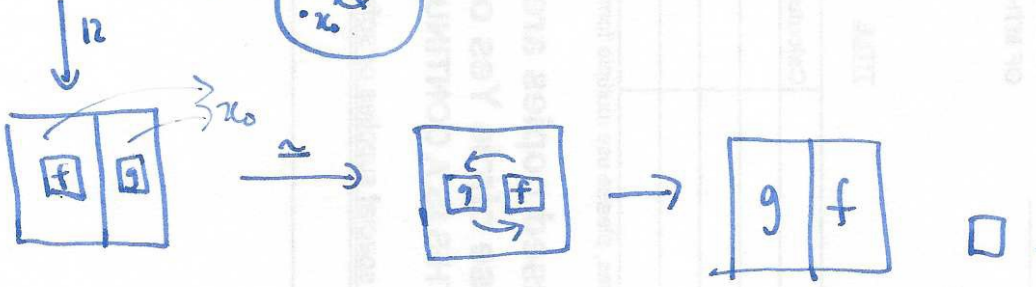


Prop^n π_n is abelian for $n \geqslant 2$.

Proof π_n = homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$.



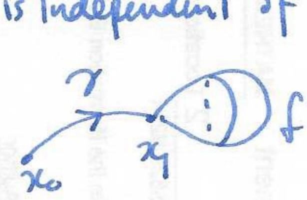
$$([f]+[g])(s_1, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \dots, s_n) & \frac{1}{2} \leq s_1 \leq 1 \end{cases}$$



$\square$.

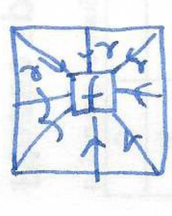
Prop^n If X is path connected then $\pi_n(X, x_0)$ is independent of x_0.

Proof Let γ be a path from x_0 to x_1



$f : (I, \partial I) \to (X, x_1)$

$\gamma f : (I, \partial I) \to (X, x_0)$



induces an isomorphism

$\pi_1(X, x_1) \to \pi_1(X, x_0)$: