M^n R-orientable, but not nec. compact. Define:

$$D_M : H^k_c(M) \longrightarrow H_{n-k}(M) \qquad (R \text{ coeffs})$$

as follows: if $\underset{\text{compact}}{K} \subset M$, then there is $\mu_K \in H_n(M|K)$ s.t.

$\mu_K =$ preferred orientation in $H_n(M|x)$ for all $x \in K$.

so $$D_K : H^k(M|K) \longrightarrow H_{n-k}(M)$$
$$\phi \longmapsto \mu_K \cap \phi$$

check: this is well defined: spose $c \in C_n(M, M\setminus K)$ represents μ_K

then $\partial c \subseteq M\setminus K$, where ϕ is 0.

so: $$\partial(c \cap \phi) = (-1)^k(\underbrace{\partial c \cap \phi}_{=0} - c \cap \underbrace{\delta\phi}_{=0}) = 0 \in C_{n-k}(M)$$

$\Rightarrow$ $\mu_K \cap \phi \in H_{n-k}(M)$ as required.

now if $\underset{\text{compact}}{K} \subset \underset{\text{compact}}{L} \subset M$ then:

$$\begin{tikzcd} H^k(M|K) \arrow[r] \arrow[d, "i^*"] & H_{n-k}(M) \arrow[d, "\cong"] \\ H^k(M|L) \arrow[r] & H_{n-k}(M) \end{tikzcd}$$ commutes as $\cap$ natural.

so now take direct limits over compact sets, to get $D : H^k_c(M) \to H_{n-k}(M)$.

Thm (Poincaré duality) M^n R-oriented manifold, not nec. compact.

$D_M : H^k_c(M;R) \longrightarrow H_{n-k}(M;R)$ is an isomorphism.

"$\phi \longmapsto \mu_K \cap \phi$" $\leftarrow$ makes sense as $\phi \in H^k_c \Rightarrow \exists\, K$ etc...

Proof Induction.

(A) $\underset{\text{open}}{U, V \subseteq M}$. If D_U, D_V and $D_{U\cap V}$ are isomorphisms, then so is $D_{U\cup V}$.

(B) Suppose M is an increasing union $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$, where each D_{U_i} is an isomorphism. Then D_M is an isomorphism.

<u>Base cases</u> ① $D_{\mathbb{R}^n}$ is an isomorphism.

② if $\underset{\text{open}}{U \subseteq \mathbb{R}^n}$, then D_U is an isomorphism.

<u>Proof</u> (assuming above)
M^n manifold, 2nd countable $\Rightarrow$ M is a countable union of open sets V_j with $V_j \underset{\text{homeomorphic}}{\cong}$ to an open set in $\mathbb{R}^n$. Set $U_i = \bigcup_{j=1}^{i} V_j$.

①, ② $\Rightarrow$ D_{V_j} and $D_{V_j \cap V_\ell}$ are isomorphisms.

Ⓐ $\Rightarrow$ D_{U_i} iso $\Rightarrow$ $D_{U_{i+1}}$ iso.

Ⓑ $\Rightarrow$ D_M iso. $\square$.

<u>Proof</u> Ⓑ

$$H^k_c(U_i) \overset{\text{last time}}{\cong} \varinjlim_{\substack{K \subseteq U_i \\ \text{compact}}} H^k(U_i|K) \overset{\text{excision}}{\cong} \varinjlim_{\substack{K \subset U_i \\ \text{compact}}} H^k(M|K)$$

$$\downarrow$$

$$H^k_c(U_{i+1}) \cong \varinjlim_{\substack{K \subseteq U_i \\ \text{compact}}} H^k(U_i|K) \leftarrow \text{this second limit contains all the } \underset{\text{compact}}{K} \subset U_i.$$

<u>note</u>: $$\varinjlim_i H^k_c(U_i) \cong \varinjlim_{\substack{K \subset M \\ \text{compact}}} H^k(M|K) \underset{\uparrow \text{ by defn.}}{\cong} H^k_c(M).$$

each $D_{U_i} : H^k_c(U_i) \to H_{n-k}(U_i)$ is an isomorphism.

so $$\underbrace{\varinjlim_i D_{U_i}}_{= D_M} : H^k_c(M) \underset{\substack{\uparrow \\ \text{direct limit of isos is an iso.}}}{\xrightarrow{\cong}} \varinjlim_i H_{n-k}(U_i) = H_{n-k}(M)$$

so $D_M : H^k_c(M) \to H_{n-k}(M)$ is an isomorphism. $\square$.

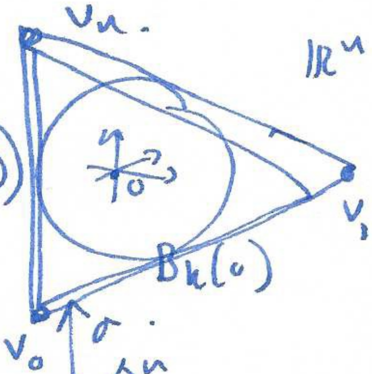
<u>Proof</u> ① $D_{\mathbb{R}^n} : H^n_c(\mathbb{R}^n) \to H_0(\mathbb{R}^n)$ is an isomorphism.

let ϕ be a generator of $H^n(\mathbb{R}^n \mid \overline{B_k(0)}) \cong \mathbb{Z}$.

and let $\sigma : \Delta^n \to \mathbb{R}^n$ represent $\mu_K \in H_n(\mathbb{R}^n \mid \overline{B_k(0)})$

then $\phi(\sigma) = +1$ (wlog).

and so $\mu_K \frown \phi = (+1)[v_n]$ which generates $H_0(\mathbb{R}^n)$ □.



<u>Proof (sketch)</u> ② Any $U \subset \mathbb{R}^n$ is a countable union of convex open sets V_j (e.g. open balls)

set $U_i = \bigcup_{j=1}^{i} V_j$. Induction (on union of $i+1$ convex open sets in $\mathbb{R}^n$)

use Ⓐ to show D_{U_i} is an iso and Ⓑ to show D_U is. □

<u>Proof</u> Ⓐ (sketch) The following diagram commutes up to sign.

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"'] & H^k_c(U)\oplus H^k_c(V) \arrow[r] \arrow[d, "D_U\oplus D_V"] & H^k_c(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & H^{k+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U\cup V) \arrow[r] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

<u>Note</u>: arrows backwards on top row!! bottom row ordinary Mayer-Vietoris

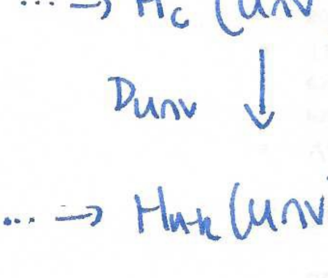
<u>top row</u>: $K \subseteq U$ compact $L \subseteq V$ compact

then: $\cdots \to H^k(U\cap V \mid K\cap L) \to H^k(U\mid K) \oplus H^k(V\mid L) \to H^k(U\cup V \mid K\cup L) \to \cdots$

excision

$(M, M\setminus K\cap L)$ $(M, M\setminus K)$ $(M, M\setminus L)$ $(M, M\setminus K\cup L)$

is exact



exact by Mayer-Vietoris (cohomology version).

direct limits of exact sequences are exact,

so top row is exact.

vertical maps are direct limits. (almost) Commutativity and 5 Lemma ⇒ result □

see Hatcher Lemma 3.36. Two follow from chain commutativity
third: barycentric subdivision and formulas for δ ...