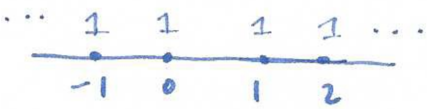
ker δ_0 in C^0 generated by $\phi: v_i \to 1$ $\supseteq$ $\ker \delta_0 \big|_{C^0_c} = \{0\}$ $\Rightarrow$ $H^0_c(\mathbb{R}) = 0$

im δ_0 in C^1 is everything $\supseteq$ $\operatorname{im} \delta_0 \big|_{C^1_c}$ $\Rightarrow$ $H^1_c(\mathbb{R}) = \mathbb{Z}$.

only contains things with sum $= 0$

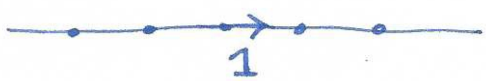
<u>Examples</u>



$\in C^0$ but not C^0_c

$\in C^0_c$ but $\delta\phi \neq 0$:

$\xrightarrow{\delta}$



$\leftarrow$ not in $\operatorname{im}(\delta)$!.

given $\phi, \psi \in C^1_c$ then $\sum_{i} \phi(v_i)$, $\sum_i \psi(v_i)$ finite.

and if $\sum \phi(v_i) = \sum \psi(v_i)$ then $\sum (\phi - \psi)(v_i) = 0$ so $\phi - \psi \in \operatorname{im} \delta_0$

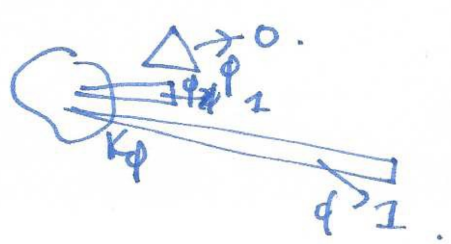
so $H^1_c(\mathbb{R}) \cong \mathbb{Z}$.

<u>note</u>: Poincaré duality holds for H^k_c !

<u>Singular</u> $C^i_c(X) = \{ \phi \in C^i(X) \mid$ there is a compact set $K_\phi \subset X$ with $\phi(\sigma) = 0$ for all $\sigma: \Delta_k \to X \setminus K \}$.

subcomplex of $C^i(X)$ so gives $H^i_c(X)$

<u>Remark</u> can't say ϕ is non-zero only on simplices contained in K_ϕ — not a subcomplex!



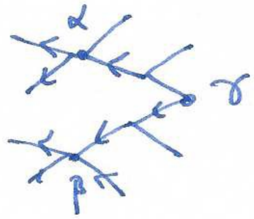
Aim $H^i_c(X) = \varinjlim H^i(X|K)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

direct limit. $\qquad H^i(X, X\setminus K)$

Direct limits

$I =$ partially ordered set, s.t. for all $\alpha, \beta \in I$, there is γ with $\begin{matrix}\alpha \le \gamma \\ \beta \le \gamma\end{matrix}$



← directed graph with no directed cycles.
s.t. any $\alpha\,\beta$ below some other vertex.

Groups G_α for each $\alpha \in I$

$f_{\alpha\beta} : G_\alpha \to G_\beta$ for each $\alpha \le \beta$.

consistency: if $\alpha \le \beta \le \gamma$ then $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$

this is called a <u>directed system of groups</u>.

Direct limit

$$\varinjlim G_\alpha = \coprod_\alpha G_\alpha \Big/ \sim$$

$a \in G_\alpha \sim b \in G_\beta$.
if there is a γ with $\begin{matrix}\alpha \le \gamma \\ \beta \le \gamma\end{matrix}$
and $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$ in G_γ.

exercise: check this is an equivalence relation. **exercise**: if $G_i \xrightarrow{\cong} G$ then $\varinjlim G_i \xrightarrow{\cong} G$.
exercise: direct limit of exact sequences exact

~~equivalently~~ can then say: $[a] + [b] = [f_{\alpha\gamma}(a) + f_{\beta\gamma}(b)]$ $\quad \begin{matrix}\alpha \le \gamma \\ \beta \le \gamma.\end{matrix}$

[alternatively: $\bigoplus_\alpha G_\alpha \big/$ subgroup gen by $a - f_{\alpha\beta}(a)$ for all $a \in G_\alpha$, $\beta \ge \alpha$.

example $I = (\mathbb{N}, \le)$

$G_\alpha = \mathbb{Z}^\alpha$

$f_{\alpha\beta} : \mathbb{Z}^\alpha \to \mathbb{Z}^\beta$

$(x_1, \dots, x_\alpha) \mapsto (x_1, \dots, x_\alpha, 0, \dots, 0)$

$\varinjlim \mathbb{Z}^\alpha = \bigoplus_{n=1}^{\infty} \mathbb{Z}$

← note $(1, 1, \dots) \notin \varinjlim G$!

$I = (\mathbb{N}, \le)$ $\quad G_n = \mathbb{Z}/2^n\mathbb{Z}$.

$G_n \to G_{n+1}$
$1 \mapsto 2$

$\varinjlim \mathbb{Z}/2^n\mathbb{Z} = \{z \in \mathbb{C} \mid$ roots of unity of order $2^k\}$.