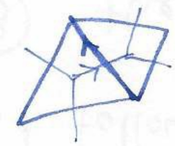
· $\cancel{\partial D(\sigma)}$ $\dot{D}(\sigma) = \bigcup \{ D(\tau) \mid \tau \gneq \sigma \}$.

so $\partial \mathring{D}(\sigma) = \dot{D}(\sigma) = \bigcup \{ D(\tau) \mid \tau > \sigma \text{ and } |\tau| = |\sigma| + 1 \}$.

so diagram commutes. $\square$.

Claim : can get proof to work with $\mathbb{Z}$ coefficients if we orient everything (carefully). $\square$.

Cap product on homology (intersection form) ($\mathbb{Z}_2$ coefficients).



$$\begin{array}{ccccc} H_k(M) & \times & H_{n-k}(M) & \longrightarrow & \mathbb{Z}_2 \\ \alpha & & \beta & & \alpha \cap \beta \end{array}$$

if $c \in C_k$ and $d \in D_{n-k}$ we have $\alpha \cap \beta = \#(c \cap d) \bmod 2$.

Fact : this turns out to be well defined and makes sense over $\mathbb{Z}$. $\square$.

Thm M closed connected with PL triangulation T. Then $\cap$ is a non-degenerate bilinear form on $H_k(M) \times H_{n-k}(M)$

Proof Pick $[\phi]$ in $H^k(M)$ with $\phi \in C^k$ and $\phi(c) = 1$.

Then $c \cap \underbrace{D(\phi)}_{\in H_{n-k}} = \phi(c) = 1$ $\square$.

Also, if η and ψ are the Poincaré duals of α and β, then you can check that $\alpha \cap \beta = (\eta \cup \psi)[M]$

Note : for general M, once you have Poincaré duality, can use this to define $\cap$ $\square$.

Thm $H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha] / \alpha^{n+1} = 0$ $\quad |\alpha| = 1$.

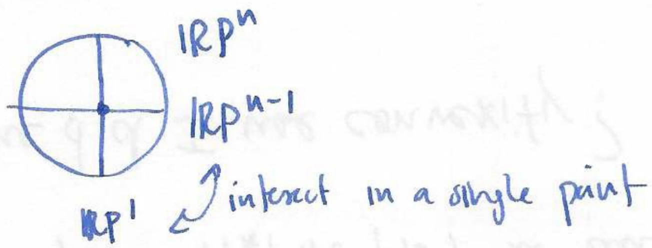
Proof $H^k(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2 = H_{n-k}(\mathbb{R}P^n; \mathbb{Z}_2)$ for all k, and 0 otherwise.

also $i : \mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1}$ induces $i^* : H^*(\mathbb{R}P^{n+1}; \mathbb{Z}_2) \longrightarrow H^*(\mathbb{R}P^n; \mathbb{Z}_2)$

which is an iso on $H^k(\mathbb{RP}^{n+1};\mathbb{Z}_2) \to H^k(\mathbb{RP}^n;\mathbb{Z}_2)$ for $k \le n$, so this determines cup products in $H_k(\mathbb{RP}^{n+1};\mathbb{Z}_2)$ for $k \le n$.

Final case: $\underset{\in H^k}{\alpha^k} \cup \underset{\in H^{n+1-k}}{\beta} \neq 0 \Rightarrow \beta = \alpha^{n-k+1}$, as required. $\square$.

Note: intersections:



Summary:

Thm M^n closed connected, PL triangulation T. Then

① $H^k(M;\mathbb{Z}_2) \cong H_{n-k}(M;\mathbb{Z}_2)$

② $\underset{\alpha}{H_k(M;\mathbb{Z}_2)} \times \underset{\beta}{H_{n-k}(M;\mathbb{Z}_2)} \to \underset{\alpha\cap\beta}{\mathbb{Z}_2}$ is non-degenerate

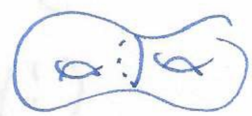
can define this geometrically, or as

$\alpha \cap \beta = (D^{-1}(\alpha) \cup D^{-1}(\beta))[M]$, where $D: H^*(M) \xrightarrow{\cong} H_*(M)$, $\varphi \mapsto [M] \cap \varphi$.

want proof of PD without triangulations.

problems: ① PD fails for $\mathbb{R}^n$: $\underset{\mathbb{Z}}{H^0(\mathbb{R}^n;\mathbb{Z}_2)} \neq \underset{0}{H_n(\mathbb{R}^n;\mathbb{Z})}$

② can't subdivide closed manifolds into closed manifolds:



Cohomology with compact supports.

simplicial version X Δ-complex, locally finite.

$C^i_c(X) =$ cochains taking non-zero values on only finitely many simplices.

subcomplex of $C^i(X)$.

Example $\mathbb{R}$