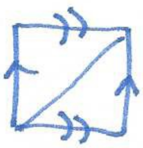
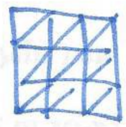
~~Idea~~ a Δ-complex is <u>simplicial</u> if any subset of vertices $\{v_0, \dots, v_n\}$ in $X^{(0)}$ are the vertices of <u>at most</u> one simplex in X



Δ-complex not simplicial.

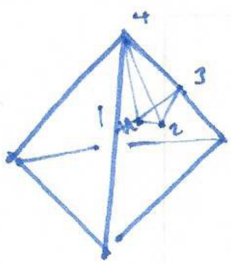


Δ-complex simplicial. (aka simplicial complex)

<u>Lemma</u> Any Δ-complex can be made simplicial by subdividing twice. $\square$

Let T be a simplicial complex structure on M^n consisting of Δ^n's with faces pairwise identified.

Each simplex σ in $S(T)$ has vertices which we order $[\hat{\alpha}_{i_1}, \hat{\alpha}_{i_2}, \dots, \hat{\alpha}_{i_k}]$ where $\alpha_{i_1} > \alpha_{i_2} > \dots > \alpha_{i_k}$

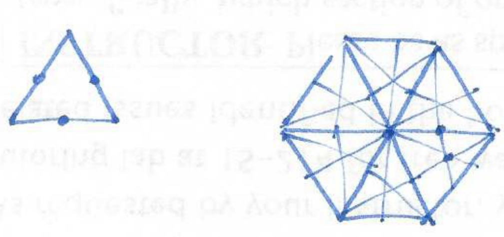


For $\sigma \in T$, define $D(\sigma) = \bigcup \{\text{int}(\alpha) \mid \alpha \subset S(T) \text{ with } \hat{\sigma} \text{ as the last vertex}\}$.

$\overline{D}(\sigma) = \text{closure of } D(\sigma) = \bigcup \{\alpha \subset S(T) \mid \text{last vertex is } \hat{\sigma}\}$.

$\dot{D}(\sigma) = \overline{D}(\sigma) \setminus D(\sigma)$

<u>Example</u>



<u>Note</u>: $\overline{D}(\sigma) = \text{cone over } \dot{D}(\sigma)$

<u>Lemma</u> a) The $D(\sigma)$ are disjoint and their union is M

b) $\overline{D}(\sigma)$ is a subcomplex of $S(T)$ with dim $n-k$, $|\sigma| = k$.

c) $\dot{D}(\sigma) = \{D(\tau) \mid \tau \supsetneq \sigma\}$.

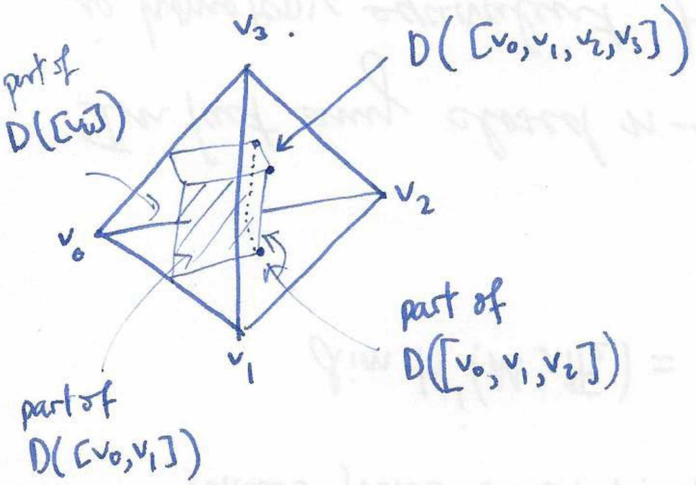
<u>Proof</u> a) every α is $S(T)$ has a unique last vertex.

b) if $|\sigma| = k$, then in some Δ^n of T, and $\alpha \subseteq \overline{D}(\sigma)$ can have at most $n-k+1$ vertices, and hence dim $\leq n-k$.

c) if $\alpha \subseteq \bar{D}(\sigma) - D(\sigma)$, let $\beta \subseteq sd(T)$ have last vertex $\hat{\sigma}$, and $\alpha < \beta$. As $\alpha \notin D(\sigma)$, α has last vertex $\hat{\tau}$ for some $\tau \subseteq T$ distinct from σ. Recall: $\beta : [\hat{\beta}_{i_1}, \hat{\beta}_{i_2}, \dots, \hat{\beta}_{i_k}]$

$\Rightarrow \tau > \sigma$ as required. □.

$$\beta_{i_1} > \beta_{i_2} > \dots > \beta_{i_k}$$
$$\sigma$$



<u>Defn</u>: T is PL if every $\bar{D}(\sigma)$ is homeomorphic to a ball in $D^{n-|\sigma|}$.

(simple Poincaré duality; $\mathbb{Z}_2$ coeffs).

<u>Thm</u>: M compact, connected n-manifold with PL triangulation T.
Then $H^k(M;\mathbb{Z}_2) \cong H_{n-k}(M;\mathbb{Z}_2)$.

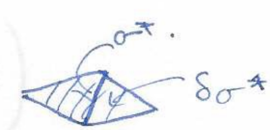
<u>Proof</u>

cochains of T $\qquad\qquad$ chains of D.

$$\begin{tikzcd} C^{k+1} \arrow[r, "\simeq"] & D_{n-k-1} \\ C^k \arrow[u, "\delta"] \arrow[r, "\simeq"] & D_{n-k} \arrow[u, "\partial"'] \end{tikzcd}$$

can think of D_{n-k} as a union of $n-k$ cells of D (as coeffs in $\mathbb{Z}_2$)
same for C^k as unions of k cells of T.
so for $\sigma_k \in T$, have both $\sigma_k^* \in C^k$ and $D(\sigma_k) \in D_{n-k}$,
this gives horizontal isomorphisms.

<u>Check</u>: diagram commutes.



- $\delta\sigma^* = \bigcup \{\tau \mid |\tau| = |\sigma|+1,\ \tau > \sigma\}$