<u>Proof</u> $\chi(M) = \sum (-1)^i \operatorname{rank}(H_i(M;\mathbb{Z}))$

$\quad\quad\;\;\; = \sum (-1)^i \dim(H_i(M;\mathbb{Z}/2))$ (*)

n odd, then i and $n-i$ have opposite parity, so cancel in pairs.

(*): · If M is a finite cell complex, then both are $\sum(-1)^i(\#\text{ of } i\text{ cells})$

· if M has finitely generated $H_*(M;\mathbb{Z})$ then use UCT:

$$\dim H^i(M;\mathbb{Z}_2) = \operatorname{rank} H_i(M;\mathbb{Z}) + \operatorname{rank}\ 2\text{-part of } H_i(M;\mathbb{Z}) + \operatorname{rank}\ 2\text{-part of } H_{i-1}(M;\mathbb{Z})$$

<u>Fact</u>: any closed n-manifold is homotopy equivalent to a CW-complex (in fact if $n \neq 4$ then M^n has a CW-complex structure). (Hatcher A.12)

<u>Dual triangulations</u>

Let M be a manifold with a triangulation T, then there is a dual cell decomposition D

k-simplex in T		$n-k$ cell in D
σ	$\longleftrightarrow$	$\hat{\sigma}$

reverses inclusion relations: $\sigma_0 < \sigma_1 \Rightarrow D(\sigma_0) > D(\sigma_1)$

$\uparrow$ subsimplex/cell

<u>aim</u>: get an isomorphism of chain complexes:

$$\underset{\text{cohomology complex of } T}{C^*} \longrightarrow \underset{\text{homology complex of } D.}{D_*}$$

this proves Poincaré duality for triangulated manifolds.

simplex $[v_0, \dots, v_n] \subset \mathbb{R}^m$. barycenter is $\hat{\Delta} = \frac{1}{n+1}\sum v_i$

recall: barycentric subdivision

$$S(\Delta) = \bigcup \left\{ [\hat{\Delta}, w_0, \dots, w_{n-1}] \;\middle|\; [w_0, \dots, w_{n-1}] \text{ simplex in } S(\partial\Delta) \right\}$$