Thm M^n closed connected. Then

a) $H_k(M;\mathbb{Z})=0$ for all $k>n$

b) if M is orientable, then $H_n(M;\mathbb{Z}) \longrightarrow H_n(M|x;\mathbb{Z})$ is an isomorphism for all $x\in M$. In particular, $H_n(M;\mathbb{Z})\cong\mathbb{Z}$.

c) otherwise $H_n(M;\mathbb{Z})=0$

Lemma A compact $\subseteq M^n$ then

1) if $x\mapsto \alpha_x$ is a section of $M_{\mathbb{Z}}\to M$ then there is an unique $\alpha_A\in H_n(M|A)$ whose image in $H_n(M|x)$ is α_x for all $x\in A$.

2) $H_k(M|A)=0$ for all $k>n$.

Lemma $\Rightarrow$ Thm 2) $\Rightarrow$ a) with $A = M$.

Let $\Gamma(M)$ be the set of sections of $M_{\mathbb{Z}}\to M$, which is a $\mathbb{Z}$-module.

There is a homomorphism $H_n(M)\longrightarrow \Gamma(M)$

$$\alpha \longmapsto \big(x\mapsto i(\alpha)\in H_n(M|x)\big)$$

by 1) this is an isomorphism

As M connected a section is determined by its value at some fixed point $p\in M$.

When M is orientable, $M_{\mathbb{Z}} = M\times\mathbb{Z}$ is an isomorphism

M non-orientable, the *only* section is the zero section. So $H_n(M)=0$.

~~Lemma~~ (Proof (of Lemma) (sketch))

assume: ① (key) true for $A, B, A\cap B \Rightarrow$ true for $A\cup B$ (Mayer-Vietoris)

② suffices to consider $M=\mathbb{R}^n$

③ holds for convex $A\subseteq\mathbb{R}^n$ hence union of convex sets (convex deformation retracts onto ball)

④ step 3 $\Rightarrow$ holds for all compact $\subseteq\mathbb{R}^n$. $\square$

Remark for any ring R can consider

$$M_R = \{\alpha_x\in H_n(M|x) \mid x\in M\} \text{ as a cover of } M$$

gives notion of R-orientability, i.e. any manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable.

Poincaré duality

Thm M^n closed, connected orientable: $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$

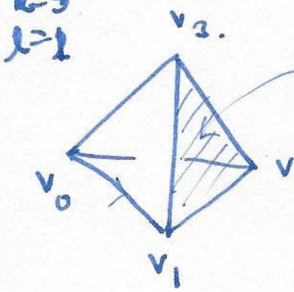
Cap product X space, R coefficient ring

$$\cap : C_k(X) \times C^\ell(X) \longrightarrow C_{k-\ell}(X) \qquad \boxed{k \geq \ell}$$

$$\sigma : \Delta^k \to X \in \qquad \phi \in$$

$$\sigma \cap \phi = \phi\left(\sigma|_{[v_0,\dots,v_\ell]}\right)\sigma|_{[v_\ell,\dots,v_k]}$$

$k=3$
$\ell=1$



$\phi([v_0,v_1])$ copies of $[v_1,v_2,v_3]$.

Exercise: $\partial(\sigma \cap \phi) = (-1)^\ell(\partial\sigma \cap \phi - \sigma \cap \delta\phi)$.

gives an R-bilinear map $H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)$.

Naturality (induced maps): $X \xrightarrow{f} Y \quad \alpha \in H_k(X) \quad \phi \in H^\ell(Y)$

then $f_*(\alpha) \cap \phi = f_*(\alpha \cap f^*(\phi))$, i.e.

$$\begin{array}{ccccc} \alpha & & f^*(\phi) & & \\ H_k(X) & \times & H^\ell(X) & \longrightarrow & H_{k-\ell}(X) \\ f_* \downarrow & & \uparrow f^* & & \downarrow f_* \\ H_k(Y) & \times & H^\ell(Y) & \longrightarrow & H_{k-\ell}(Y) \\ f_*(\alpha) & & \phi & & \end{array}$$

commutes as best it can.

Thm (Poincaré duality) M^n closed connected R-orientable $[M] \in H_n(M;R)$ a generator. Then $D : H^k(M) \longrightarrow H_{n-k}(M)$, $\phi \longmapsto [M] \cap \phi$ is an isomorphism.

Corollary M^n closed connected. Then $H_k(M;\mathbb{Z}_2) \cong H_{n-k}(M;\mathbb{Z}_2)$

(as $H^k(M;\mathbb{Z}_2) \cong H_k(M;\mathbb{Z}_2)$)

Corollary If n odd then $\chi(M^n) = 0$, M closed connected