<u>Problem</u>: not all compact n-manifolds have triangulations [... Manolescu '13].
(all <u>smooth</u> manifolds have triangulations though).

<u>Orientations</u> (preserved under rotation, reversed by reflection)

<u>Def$^{\underline{n}}$</u> An orientation of $\mathbb{R}^n$ at x is a choice of <u>generator</u> μ in

$$H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\};\mathbb{Z}) \underset{\uparrow\,\text{LES pair}}{\cong} H_{n-1}(\mathbb{R}^n\setminus\{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

Suppose B open ball with $x \in B$

(cut set!)

$$\begin{tikzcd} H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{x\}) & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus B) \arrow[l, "\cong"', "i_*"] \arrow[d, "\cong", "i_*"'] \arrow[r, "\cong", "\text{excision}"'] & H_n(S^n = \mathbb{R}^n/\mathbb{R}^n\setminus B) \\ & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{y\}) & \end{tikzcd}$$

<u>Def$^{\underline{n}}$</u> The <u>local homology of X at A</u> is $H_n(X|A) = H_n(X, X\setminus A)$.

If M is an n-manifold, then $H_n(M|x) \underset{\uparrow\,\text{excision}}{\cong} H_n(\mathbb{R}^n|\{pt\})$.

a <u>local orientation</u> at $x \in M$ is a choice of generator of $H_n(M|x) \cong \mathbb{Z}$.

<u>Def$^{\underline{n}}$</u> An <u>orientation</u> of M is a function $x \longmapsto \mu_x$ such that for all open sets $U \cong \mathbb{R}^n$ and bounded open balls $B \subset U$, there is $\mu \in H_n(M|B)$ such that

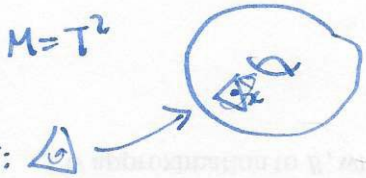
$$\begin{array}{ccccccc} H_n(M|x) & \longleftarrow & H_n(M|B) & \cong & H_n(U|B) & \cong & \mathbb{Z} \quad \text{for all } x \in B. \\ \mu_x & \longleftarrow\!\shortmid & \mu & & & & \end{array}$$

If an orientation exists, M is called <u>orientable</u>.

<u>Thm</u> M closed connected n-manifold, if M is orientable then $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and $H_n(M;\mathbb{Z}) \to H_n(M|x;\mathbb{Z})$ is an isomorphism for all $x \in M$.
Otherwise, $H_n(M;\mathbb{Z}) = 0$

<u>Remarks</u>: ① $H_n(M;\mathbb{Z}) \xrightarrow{\cong} H_n(M|x;\mathbb{Z})$ iso for all x $\Rightarrow$ orientability:
fix a generator μ of $H_n(M;\mathbb{Z})$ and set μ_x = image in $H_n(M|x;\mathbb{Z})$.

② all manifolds are $\mathbb{Z}/2\mathbb{Z}$ orientable, as $H_n(M|x;\mathbb{Z}_2)$ has a single non-zero element.

③ Example $M=T^2$  generator $[\sigma]$ of $H_n(M|x) = H_n(M, M\setminus\{x\})$ is: $\sigma:\Delta^2\to T^2$.

$\sigma:$

Motivation $\mathbb{RP}^2$ non-orientable, has orientable double cover $\begin{matrix} S^2 \\ \downarrow \\ \mathbb{RP}^2 \end{matrix}$ 

Defn M n-manifold, $\tilde M = \{\mu_x \mid x\in M,\ \mu_x \text{ a local orientation at } x\}$.

topology: for all $\underset{\text{ball}}{\text{bounded}} \subseteq \mathbb{R}^n \underset{\text{open}}{\subseteq} M$ and generator $\mu_B \in H_n(M|B)$

then $U(\mu_B) = \{\mu_x\in\tilde M \mid x\in B,\ \underset{H_n(M|B)\to H_n(M|x)}{\mu_B \mapsto \mu_x}\}$ are open sets.

projection $p:\tilde M\to M$, $\mu_x\mapsto x$. check $p:\tilde M\to M$ is a covering space of degree 2.

Propn $\tilde M$ is orientable, with orientation

$$\tilde\mu_{\mu_x} \in H_n(\tilde M|\mu_x) \cong H_n(U(\mu_B)|\mu_x) \xrightarrow[\cong]{p_*} H_n(B|x) \ni \mu_x$$

Call $\tilde M$ the orientable double cover.

Propn Suppose M is connected, then M is orientable iff $\tilde M$ has two connected components.

Corollary If $\pi_1 M = 1$, then M is orientable. [if $\mathrm{ab}(\pi_1 M)=1$!] (e.g. S^n, $\mathbb{CP}^n$).

We can define a larger covering space

$$M_{\mathbb{Z}} = \{\alpha_x\in H_n(M|x) \mid x\in M\} \supseteq \tilde M \quad \text{(can define this for any ring).}$$

$$\begin{matrix} M_{\mathbb{Z}} \\ s \uparrow \downarrow p \\ M \end{matrix}$$

For orientable M, this is $M\times\mathbb{Z}$. A section of $M_{\mathbb{Z}}\xrightarrow{p} M$ is a continuous map $s: M\to M_{\mathbb{Z}}$ where $p\circ s = \mathrm{id}_M$.

Example $s: x\mapsto 0\in H_n(M|x)$.