Proof (simple Künneth; sketch)

fix Y, consider functors
$$h^n(X,A) = \bigoplus_i \left(H^i(X,A;R) \otimes_R H^{n-i}(Y;R) \right)$$
$$k^n(X,A) = H^n(X\times Y, A\times Y; R)$$

relative cross product $\mu: h^n(X,A) \to k^n(X,A)$

want: this is an isomorphism when $A=\phi$

show: · h^*, k^* cohomology theories

· μ natural transformation (commutes with induced homomorphisms and coboundary map in l.e.s. of pair).

Prop: If μ is an iso for $(\{pt\}, \phi)$ then it is an iso for all pairs $\square$.

check $X=\{pt\}$, $A=\phi$ $\square$

Application

Thm (Hopf) If $\mathbb{R}^n$ has a division algebra structure over $\mathbb{R}$, then n is a power of 2.

Recall: algebra: bilinear map $\mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$
$(a,b) \longmapsto ab$

distributive: $a(b+c) = ab+ac$
$(a+b)c = ac+bc$
$\alpha(ab) = (\alpha a)b = a(\alpha b)$

not necessarily: commutative
associative
w/identity element

division algebra: $a\neq 0$ then $ax=b$, $xa=b$ always solveable (equivalently: no zero divisors).

Examples $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ Quaternions: $\langle 1,i,j,k \rangle$ $ij=k$
$\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} \cong \mathbb{R}^4$
$i^2=j^2=-1$
$ji=-ij$

$\mathbb{O}$ Octonions (not associative)
$\mathbb{O} \cong \mathbb{R}^8$

Thm [Bott-Milnor] [Kervaire] That's all $\square$.

Thm (Hopf) $\mathbb{R}^n$ division algebra, then $n = 2^m$.

Proof $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ gives $g : S^{n-1} \times S^{n-1} \to S^{n-1}$

$$(x,y) \mapsto \frac{xy}{|xy|}$$

rmk: $-(xy) = (-x)y = x(-y)$ so $g(-x,y) = -g(x,y) = g(x,-y)$

so we get $h : \mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}P^n$

$\mathbb{Z}_2$ coeffs: $h^* : H^1(P^n \times P^n) \xleftarrow{h^*} H^1(P^n) \cong \mathbb{Z}_2[\gamma]$

claim: $\alpha + \beta \longleftarrow\!\shortmid \gamma$ where $\alpha = p_1^*(\gamma)$ $\beta = p_2^*(\gamma)$.

rmk: cup product structure $\Rightarrow$ this completely determines h^* on H^k.

proof (of claim): $(n > 1)$ $\pi_1 P^n = \mathbb{Z}/2\mathbb{Z}$

find: $\pi_1(P^n \times P^n) \xrightarrow{h_*} \pi_1 P^n$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$

S^n λ generator of $\pi_1 P^n$

what is $h_*(1,0)$? $(1,0) = (\lambda, \text{const } x_0) \longmapsto \dfrac{\lambda x_0}{|\lambda x_0|}$

i.e. image of λ under linear map $z \mapsto z x_0$,

so is still a path joining antipodal points, so $h_*(1,0) = 1$.

similarly $h_*(0,1) = 1$

$H_1(P^n) = ab(\pi_1 P^n) = \pi_1 P^n$ so same action on H_1.

$H^1(X;\mathbb{Z}_2) = \mathrm{Hom}(H_1(\ ;\mathbb{Z}_2), \mathbb{Z}_2)$ gives claim. $\square$.

now $\gamma^n = 0$ in $H^*(P^{n-1};\mathbb{Z}_2)$, so $0 = h^*(\gamma^n) = (\alpha+\beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k}$

in $H^*(P^{n-1} \times P^{n-1};\mathbb{Z}_2)$, so $\binom{d}{k} \equiv 0 \bmod 2$ for all $0 < k < d$.

equivalently, $(1+x)^n = 1 + x^n$ in $\mathbb{Z}_2[x]$. Fact: this only happens if $n = 2^m$.

write $n = d_1 + d_2 + \dots + d_\ell \leftarrow$ powers of 2 w/ $d_i < d_{i+1}$

rmk: $(1+x)^2 = 1 + x^2 \bmod 2 \Rightarrow (1+x)^{2^m} = 1 + x^{2^m}$

so $(1+x)^n = (1+x)^{d_1} \cdots (1+x)^{d_\ell} = (1+x^{d_1}) \cdots (1+x^{d_\ell}) \leftarrow$ same poly with 2^ℓ terms.

$\square$.

§3.3 Poincaré duality

Def^n An n-manifold is a Hausdorff, 2nd-countable topological space, where every point has an open neighborhood homeomorphic to $\mathbb{R}^n$.

(topological manifold; also smooth, piecewise linear manifolds).

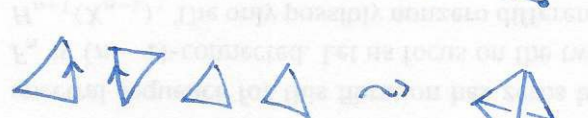
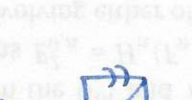
Poincaré Duality M closed (compact, $\partial M = \emptyset$) connected n-manifold, then $H_k(M;\mathbb{Z}_2) \cong H_{n-k}(M;\mathbb{Z}_2)$. If M is orientable then $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$

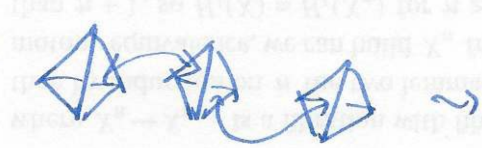
Note manifold, local condition $\Rightarrow$ global info.

Thm M closed connected n-manifold, then $H_n(M;\mathbb{Z}) = \underset{\text{orientable}}{\mathbb{Z}}$ or $\underset{\text{non-orientable}}{0}$ and $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$.

Def^n A triangulation of M is a Δ-complex structure consisting of n-simplices with their $(n-1)$-dim faces glued in pairs.

Example $n=1$ 

$n=2$:  or 

$n=3$:  $\rightsquigarrow M^3$.

Suppose M has a triangulation, then $H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$

- generator: 1 copy of every simplex
- not connected, then get $\oplus\mathbb{Z}_2$
- not compact? ($H_n(M;\mathbb{Z}_2) = 0$)

What about $H_n(M;\mathbb{Z})$? start with one simplex:  choose orientation.

choose compatible orientation on adjacent simplices 

if everything fits together, get $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$.

if not $H_n(M;\mathbb{Z}) = 0$ (e.g. Möbius band: