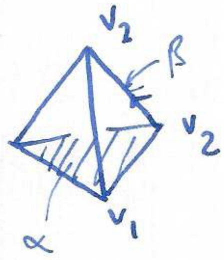
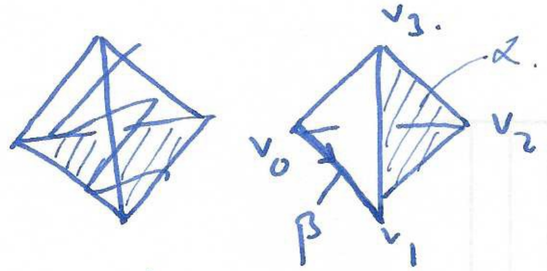
Proof (sketch)

$k=2, \ell=1$



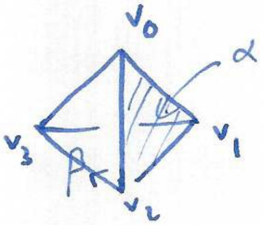
$\alpha \cup \beta(\sigma)$



$\beta \cup \alpha(\sigma)$

define $\bar{\sigma} = \sigma \circ \left([v_0, \dots, v_n] \to [v_n, \dots, v_0] \right)$

linear map which reverses order of vertices.



$\alpha \cup \beta(\bar{\sigma}) \leftarrow$ almost $\beta \cup \alpha(\sigma)$.

set $\epsilon_n = (-1)^{\frac{n(n+1)}{2}}$ (whether τ preserves/reverses orientation)

and define
$$\rho : C_n(X) \to C_n(X)$$
$$\sigma \longmapsto \epsilon_n \bar{\sigma}$$

Facts
- ρ is a chain map, chain homotopic to id.
- ρ^* on C^* induces id on $H^*(X)$
- $\epsilon_{k+\ell}\, \rho^*(\alpha \cup \beta) = \epsilon_k \epsilon_\ell\, \rho^*(\alpha) \cup \rho^*(\beta)$

$\Rightarrow$ in H^*:
$$\alpha \cup \beta = \epsilon_{k+\ell} \epsilon_k \epsilon_\ell\, \beta \cup \alpha$$
$$= (-1)^{k\ell}\, \beta \cup \alpha. \quad \square$$

Defn Exterior algebra $\Lambda_R[\alpha_1, \alpha_2, \dots]$: free R-module with basis $\alpha_{i_1} \cdots \alpha_{i_k}$ with $i_1 < \dots < i_k$ with $\alpha_i \alpha_j = -\alpha_j \alpha_i$ $\quad \alpha_i^2 = 0$.

$$H^*(T^n; R) \underset{\text{rings}}{\cong} \Lambda_R[\alpha_1, \dots, \alpha_n].$$

Künneth formulas

Q: what is $H^*(X\times Y)$?

we have
$$H^*(X)\times H^*(Y) \xrightarrow{\times} H^*(X\times Y)$$
$$(\alpha, \beta) \longmapsto \alpha\times\beta$$

bilinear not a homomorphism

however get a homomorphism if we replace $H^*(X)\times H^*(Y)$ with $H^*(X)\otimes_R H^*(Y)$

recall abelian groups A, B, then $A\otimes B$ generated by $a\otimes b$

relations $(a+a')\otimes b = a\otimes b + a'\otimes b$

$a\otimes(b+b') = a\otimes b + a\otimes b'$

note A, B abelian gps $\Rightarrow$ $\mathbb{Z}$-modules.
and $A\otimes B = A\otimes_{\mathbb{Z}} B$.

(zero: $0\otimes 0 = a\otimes 0 = 0\otimes b$)

(inverse: $-(a\otimes b) = (-a)\otimes b = a\otimes(-b)$).

A, B R-modules, then $A\otimes_R B$ is $A\otimes B / ra\otimes b = a\otimes rb$

Examples $A = \mathbb{Z}^2$ (a_1, a_2) $B = \mathbb{Z}^3$ ($b_1\, b_2\, b_3$) $A\otimes B \cong \mathbb{Z}^6$ ($a_i\otimes b_j$) $\ni 3a_1\otimes b_2 + 4a_1\otimes b_3 - 5a_2\otimes b_2$

$A\otimes B = A\otimes_R B$ if $R = \mathbb{Z}_m$ or $\mathbb{Q}$, but not equal in general.

e.g. $R = \mathbb{Q}(\sqrt{2}) \leftarrow$ 2 dim vector space over $\mathbb{Q}$

$R\otimes_{\mathbb{Q}} R \leftarrow$ 4 dim vector space over $\mathbb{Q}$
basis $1\otimes 1$ $\sqrt{2}\otimes 1$ $1\otimes\sqrt{2}$ $\sqrt{2}\otimes\sqrt{2}$

this always holds. $R\otimes_R R \cong R$.
$1\otimes\sqrt{2} = \sqrt{2}\otimes 1$.

similarly $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C} = \mathbb{C}$ but $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ dim 4.

so we get a homomorphism
$$H^*(X)\otimes H^*(Y) \xrightarrow{\times} H^*(X\times Y)$$
$$a\otimes b \longmapsto a\times b$$

define a multiplication by: $(a\otimes b)\cdot(c\otimes d) = (-1)^{|b||c|}(a\cup c)\otimes(b\cup d)$

then this map is a ring homomorphism

Thm (simple Künneth) If X, Y are CW-complexes and $H^*(Y)$ is a free R-module, then $H^*(X)\otimes H^*(Y) \xrightarrow{\times} H^*(X\times Y)$ is a ring isomorphism.

Corollary: always holds if R = field.

<u>Examples</u> $H^*(\mathbb{RP}^\infty\times\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \cong H^*(\mathbb{RP}^\infty;\mathbb{Z}/2)\otimes H^*(\mathbb{RP}^\infty;\mathbb{Z}/2)\cong \mathbb{Z}_2[\alpha]\otimes\mathbb{Z}_2[\beta]$.

$\cong \mathbb{Z}_2[\alpha,\beta]$.

$H^*(\underbrace{S^1\times\cdots\times S^1}_{T^n};R)\cong \Lambda_R[\alpha_1,\dots,\alpha_n]$

<u>Thm</u> (General homology Künneth) X,Y CW complexes, R principle ideal domain, then there are (natural) short exact sequences

$$0\to\bigoplus_i \left(H_i(X;R)\otimes_R H_{n-i}(Y;R)\right) \xrightarrow{\times} H_n(X\times Y;R)\to \bigoplus_i \mathrm{Tor}_R(H_i(X;R), H_{n-i-1}(Y;R))\to 0$$

<u>Corollary</u> if F is a field, X,Y CW complexes, then

$$\bigoplus_i H_i(X;F)\otimes_F H_{n-i}(Y;F)\xrightarrow{\times} H_n(X\times Y;F)$$ is an isomorphism for all n.

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recall $\mathrm{Tor}(A,B)$: $\cdots\to F_2\to F_1\to F_0\to A\to 0$ free resolution

$$\to F_1\otimes B\to F_0\otimes B\to A\otimes B\to 0$$
$$H_1=\mathrm{Tor}(A,B)$$

$\mathrm{Tor}_R(A,B)$ A,B R-modules $\cdots\to F_2\to F_1\to F_0\to A\to 0$ free resolution of R-modules.

$$\cdots\to F_1\otimes_R B\to F_0\otimes_R B\to A\otimes_R B\to 0$$
$$H_1=\mathrm{Tor}_R(A,B)$$

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cross products in homology: $H_i(X;R)\times H_j(Y;R)\xrightarrow[\text{bilinear}]{\times} H_{i+j}(X\times Y;R)$

can define this at the cellular chain level.

define on cells: $\overset{e^i}{[\alpha]}\quad \overset{e^j}{[\beta]} \longmapsto \overset{e^i\times e^j}{[\alpha\times\beta]}$.

extend bilinearly on chains

<u>note</u>: gives $H_i(X;R)\otimes_R H_j(Y;R)\xrightarrow{\times} H_{i+j}(X\times Y;R)$ homomorphism.

<u>Key fact</u>: $d(e^i\times e^j) = de^i\times e^j + (-1)^i e^i\times de^j$

$\Rightarrow$ cycle × cycle is a cycle
boundary × cycle is boundary
cycle × boundary is boundary

$\Rightarrow (f\times g)_* = f_*\times g_*$.