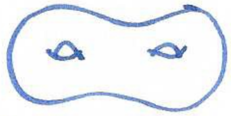
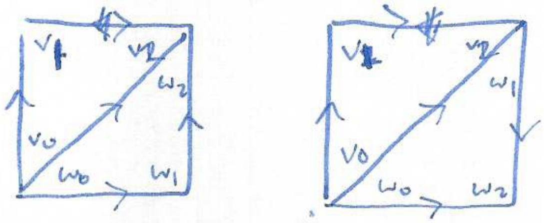
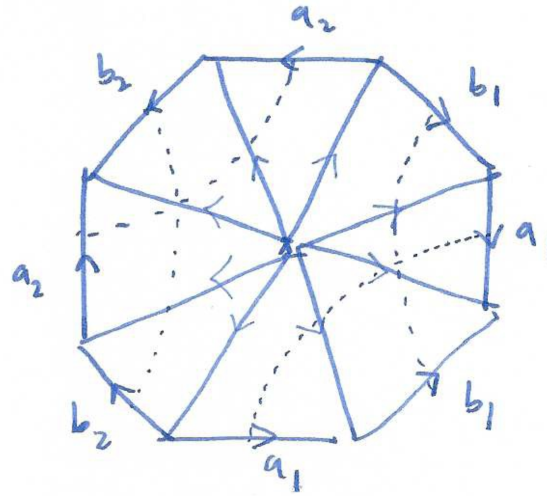
Example $S =$ 

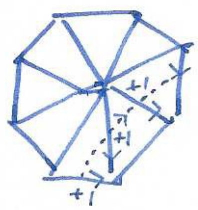
Δ-complex: order of vertices agrees with each subsimplex

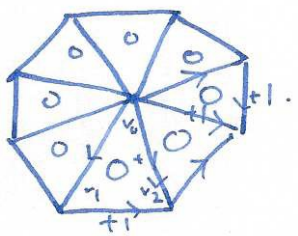
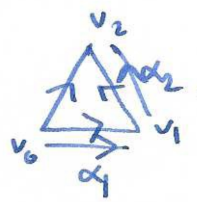




$H^1(S;\mathbb{Z}) = \mathrm{Hom}(H_1(S;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}^4 \leftarrow$ basis $\{a_1, b_1, a_2, b_2\}$.

$\alpha_1, \beta_1, \alpha_2, \beta_2$ dual basis to a_1, b_1, a_2, b_2

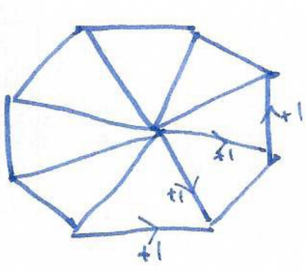
represent this as a chain:  claim $\delta\alpha_1 = 0$ (check!).

$\alpha_1 \cup \alpha_1$:  

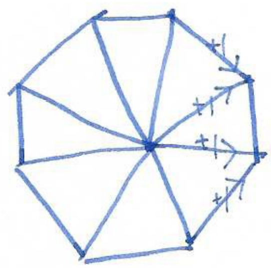
$\alpha_1(\sigma|_{[v_0,v_1]})\,\alpha_1(\sigma|_{[v_1,v_2]})$

so $\alpha_1 \cup \alpha_1 = 0$

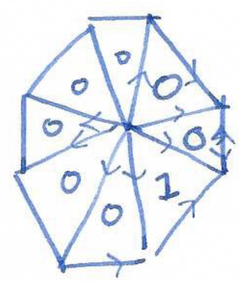
$\alpha_1 \cup \beta_1$:

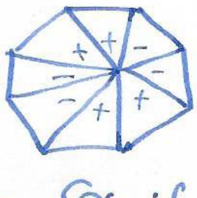


α_1



β_1

 $\alpha_1 \cup \beta_1$.

$H^2(S;\mathbb{Z}) = \mathrm{Hom}(H_2(S;\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}$ generator $[\gamma] \leftarrow$ one copy of every simplex with appropriate sign:  so $\alpha_1 \cup \beta_1(\sum \sigma_{2,i}) = +1$.

so: $\alpha_i \cup \beta_j = \begin{cases} \gamma & \text{if } i=j \\ 0 & \text{else} \end{cases}$ $(= -\beta_j \cup \alpha_i)$.

<u>Fact</u> $\quad H^1 \cong H_1 \quad$ (for S)

$\cup$ cup product $\qquad \cap$ intersection form.

<u>Relative homology</u>

$$H^k(X,A)\times H^l(X,A) \longrightarrow H^{k+l}(X,A)$$
$$H^k(X,A)\times H^l(X)$$
$$H^k(X)\times H^l(X,A)$$

because: $C^n(X,A) = \mathrm{Hom}(\underset{C_n(X)/C_n(A)}{\underset{\shortparallel}{C_n(X,A)}}, R) = \phi \in C^n(X)$ s.t. $\phi|_{C_n(A)} = 0$.

so $\quad \alpha\cup\beta(\underset{C_n(A)}{\underset{\cap}{\sigma}}) = 0$

$\subset A$.

<u>Induced homomorphisms</u> $\quad X \xrightarrow{f} Y \qquad H^*(X) \xleftarrow{f^*} \underset{\alpha,\beta}{H^*(Y)}$.

$$f^*(\alpha\cup\beta) = f^*(\alpha)\cup f^*(\beta)$$

(follows from chain level def$^{\underline{n}}$).

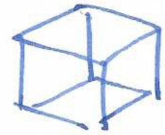
<u>Cross product</u> (external cross product)

$$\begin{array}{ccc} X\times Y & \xrightarrow{p_2} & Y \\ \downarrow p_1 & & \\ X & & \end{array}$$

$$H^k(X)\times H^l(Y) \xrightarrow{\times} H^{k+l}(X\times Y)$$
$$\alpha,\ \beta \longmapsto p_1^*(\alpha)\cup p_2^*(\beta).$$

relative version: $H^k(X,A)\times H^l(Y,B) \to H^{k+l}(X\times Y,\ A\times Y \cup X\times B)$.

Example $T^n = S^1 \times \cdots \times S^1 = \mathbb{R}^n / \mathbb{Z}^n$



projection onto the ith factor $\quad \downarrow p_i \quad S^1$

Pick $\alpha \in H^1(S^1)$ a generator, set $\alpha_i = p_i^*(\alpha)$, then

$H^*(T^n) =$ exterior algebra on $\alpha_1, \dots, \alpha_n$; $\alpha_i \cup \alpha_j = -\alpha_j \cup \alpha_i$

$H^k(T^n) =$ free R-module with basis $\alpha_{i_1} \cup \alpha_{i_2} \cup \cdots \cup \alpha_{i_k}$
$\quad i_1 < i_2 < \cdots < i_k$

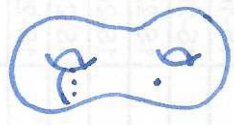
Relation to cross product:

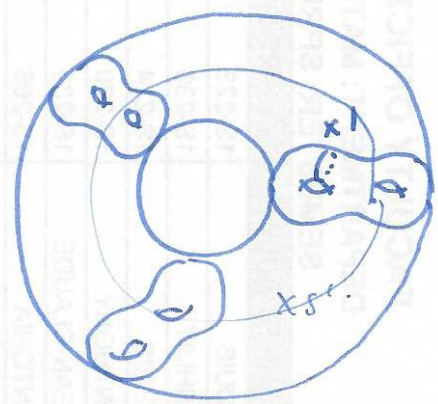
$\underline{n=4}$ $\quad \alpha_1 \cup \alpha_3 = \alpha \times 1 \times \alpha \times 1$
$\quad \alpha_1 \cup \alpha_4 = \alpha \times 1 \times 1 \times \alpha$ etc.

~~equivalently~~

Lemma $H^{n+1}(Y) \times H^{n}(Y) \longrightarrow H^{n+1}(Y \times S^1)$ is an isomorphism
$$(\beta_1, \beta_2) \longmapsto \beta_1 \times 1 + \beta_2 \times \alpha.$$

Intuition: think about homology:





Point: every cycle can be expressed as the sum of two product cycles.

Lemma $\Rightarrow$ result on $H^*(T^n)$ by induction $\square$.

Proof Consider:
$$H^{n+1}(Y) \xrightarrow[\text{projection}]{p^*} H^{n+1}(Y \times S^1) \xrightarrow[\text{inclusion}]{i^*} H^{n+1}(Y)$$
$$\beta_1 \longmapsto \beta_1 \times 1 = p^*(\beta_1) \hookrightarrow \cdot$$

is an isomorphism, as

$\Rightarrow p^*$ injective

$p^*(\beta) = \beta \times 1$ by explicit cochain representatives on product cells.

$Y \times S^1$ has cells $\quad e^{n+1} \times v \quad e^n \times e$

$\quad \to \beta(e^{n+1}) . 1 \quad \to 0.$
$\quad \beta \times 1$

(1) consider pair $(Y\times I, Y\times \partial I)$  $_{(Y\times I, Y\times\partial I)} \simeq Y\times$  $(Y\times S^1, Y\times\{0\})$.

$$\cdots \xleftarrow{0} H^{k+1}(Y\times I, Y\times \partial I) \xleftarrow{\delta}$$

$$H^k(Y\times\partial I) \leftarrow H^k(Y\times I) \xleftarrow{0} H^k(Y\times I, Y\times\partial I) \leftarrow \cdots$$

$$\begin{array}{ccc} H^k(Y\times\partial I) & \leftarrow & H^k(Y\times I) \\ \cong & & \cong \\ H^k(Y)\oplus H^k(Y) & \leftarrow & H^k(Y) \\ (\beta,\beta) & \xleftarrow{\text{inj.}} & \beta \end{array}$$

$$p_1^\#\delta\phi\cup p_2^\#\psi + p_1^\#\phi\cup p_2^\#\delta\psi, \quad p_1^\#\delta\phi = 0$$

$$p_1^\#\phi\cup p_2^\#\delta\psi \xleftarrow{\delta} p_1^\#\phi\cup p_2^\#\psi.$$

(2) so we get:

$$\begin{array}{ccccc} 0 & \leftarrow & H^{k+1}(Y\times I, Y\times\partial I) & \xleftarrow{\delta} & H^k(Y\times\partial I) \\ & & \uparrow\times & & \uparrow\times \\ & & H^k(Y)\times H^1(I,\partial I) & \xleftarrow{1\times\delta} & H^k(Y)\times H^0(\partial I) \\ & & (\phi,\ \delta\psi) & & (\phi,\ \psi) \end{array}$$

<u>claim</u>: this commutes.

so $\text{image}(\delta) = H^k(Y)\times\alpha \subset H^{k+1}(Y\times I, Y\times\partial I)$.

(3) now consider $(Y\times S^1, Y\times\{0\})$:

$$H^{k+1}(Y\times S^1, Y\times\{0\})$$

$$0 \leftarrow H^k(Y\times\{0\}) \underset{p^*}{\overset{i^*}{\rightleftarrows}} H^k(Y\times S^1) \leftarrow H^k(Y\times S^1, Y\times\{0\}) \leftarrow 0$$

$$\delta = 0, \qquad H^k(Y\times\{0\}) \cong H^k(Y), \qquad H^k(Y\times S^1, Y\times\{0\}) \cong H^{k-1}(Y)\times\alpha.$$

(4)
$$\overbrace{Y\times\{0\} \xrightarrow{i} Y\times S^1 \xrightarrow{p} Y\times\{0\}}^{1_Y} \implies \underline{i^* \text{ onto}} \implies \delta = 0.$$

$$H^k(Y\times\{0\}) \xleftarrow{i^*} H^k(Y\times S^1) \xleftarrow{p^*} H^k(Y\times\{0\}).$$

split exact sequence $\Rightarrow$ $H^k(Y\times S^1) \cong H^k(Y)\oplus H^{k-1}(Y)$.

$$\beta\times 1 \qquad \beta\times\alpha. \qquad \square$$

Other examples of explicitly known ring structures

Thm $H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]$ $\quad |\alpha| = 1$.

$H^*(\mathbb{RP}^n;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/\alpha^{n+1}$ $\quad |\alpha| = 1$

$H^*(\mathbb{CP}^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha]$ $\quad |\alpha| = 2$

$H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1}$ $\quad |\alpha| = 2$. $\quad \square$.

Wedge sums (in fact disjoint unions)

recall $f: X \to Y$ induces a ring homomorphism $f^*: H^*(Y) \to H^*(X)$

so the group isomorphism $H^*(\coprod_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha H^*(X_\alpha)$ is a ring isomorphism.

induced by inclusion maps $i_\alpha: X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$.

coordinate wise product structure.

similarly $\tilde{H}^*(\bigvee_\alpha X_\alpha) \xrightarrow{\cong} \prod_\alpha \tilde{H}^*(X_\alpha)$ is a ring ~~homom~~ isomorphism.

coordinatewise product

(as long as (X_α, x_α) good pair for each α).

can see this at cochain level:

$X \vee Y$ $\quad$ X $\quad$ Y

$\phi \quad 0$

$0 \quad \psi$.

given $\phi \in H^k(X)$ extend by zero to $\phi \in H^k(X \vee Y)$. (similarly for ψ).

then $\phi \cup \psi = 0$ on all simplices.

Example $\mathbb{CP}^2 \not\simeq S^2 \vee S^4$

$T^2 \not\simeq S^1 \vee S^1 \vee S^2$

Thm $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$ if $\alpha \in H^k(X,A;R)$
$\beta \in H^l(X,A;R)$

$H^*(X,A;R)$ called (graded) commutative.

Corollary • if k odd, then $\alpha \cup \alpha = -\alpha \cup \alpha$ i.e. $2(\alpha \cup \alpha) = 0 \in H^{2k}(X)$

So if H^{2k} has no 2-torsion then $\alpha \cup \alpha = 0$.

• if cohomology only even dims, or $R = \mathbb{Z}/2\mathbb{Z}$, then $H^*(X;R)$ commutative