Split exact sequences

Splitting lemma $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ short exact sequence

The following are equivalent:

a) there is a homomorphism $p : B \to A$ s.t. $pi = 1_A$

b) there is a homomorphism $s : B \to C$ s.t. $js = 1_C$

c) there is an isomorphism $B \cong A \oplus C$ s.t.

$$\begin{tikzcd}
 & & B \arrow[dr, "j"] \arrow[d, "\cong"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$a \mapsto (a,0)$ $\quad (a,c) \mapsto c$

commutes

Proof c) $\Rightarrow$ a) c) $\Rightarrow$ b) ✓.

a) $\Rightarrow$ c) define $B \to A \oplus C$ by $b \mapsto (p(b), j(b))$ onto.

claim: isomorphism, check injective. spose $p(b) = 0$, $\underbrace{j(b) = 0}_{\Rightarrow b \in \text{image}(i)}$

so $b = i(a)$ for some $a \in A$

$p(b) = pi(a) = a = 0$ so $b = i(0) = 0$. ✓.

b) $\Rightarrow$ c) define $A \oplus C \to B$ by $(a,c) \mapsto i(a) + s(c)$

injective: spose $i(a) + s(c) = 0$ then $ji(a) = 0 = js(c) = c \Rightarrow c = 0$

$i(a) = 0$ i injective $\Rightarrow a = 0$.

surjective: $b \in B$, then $j(b) \in C$ and $sj(b) \in B$, $b - sj(b) \in \ker(j)$

$\Rightarrow \exists\, a \in A$ s.t. $i(a) = b - sj(b)$. claim: $(a, j(b)) \mapsto b$.

check: $(a, j(b)) \mapsto i(a) + sj(b) = b - sj(b) + sj(b) = b$. $\square$.

$\text{Hom}(\mathbb{Z}^2, \mathbb{Z}) \ni \varphi, \psi$ $\quad (\varphi + \psi)(g) = \varphi(g) + \psi(g)$

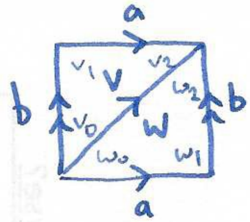
basis a, b. $\quad$ zero: $0(g) = 0$, inverse $\bar{\varphi}(g) = -\varphi(g)$

φ determined by $\varphi(a), \varphi(b)$ define: $\alpha(a) = 1$, $\alpha(b) = 0$ $\quad \beta(a) = 0$, $\beta(b) = 1$ then $\varphi(1,0) = a$, $\varphi(0,1) = b$

$\cancel{\varphi(a,m) = a\varphi(a) + m\varphi(b)}$

$\varphi = a\alpha + b\beta$.

Example T^2



$H^1(M) \times H^1(M) \longrightarrow H^2(M)$ (simplicial homology)

$\mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}$

chain complex.

$0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$

$\langle v, w \rangle \quad \langle a, b, c \rangle$.

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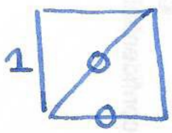
basis for $H_1(M)$ consists of edges a, b above.

$H^1(M) \cong \mathrm{Hom}(H_1(M); \mathbb{Z}) \cong \mathbb{Z}^2$ (universal coefficients)

choose dual basis α, β $\quad \alpha(a) = 1 \quad \alpha(b) = 0$

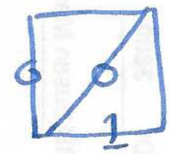
$\beta(a) = 0 \quad \beta(b) = 1$

represent α by a cocycle ϕ : intuition: take intersections with a.



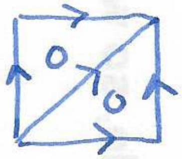
check $\delta\phi = 0$

represent β by ψ

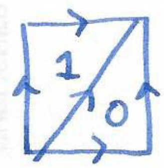


check $\delta\psi = 0$

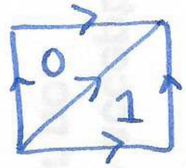
compute: $\alpha \cup \alpha$



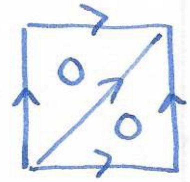
$\alpha \cup \beta$



$\beta \cup \alpha$



$\beta \cup \beta$



$H_2(M) \cong \mathbb{Z}$ generated by $V - W$

$H^2(M) \cong \mathrm{Hom}(H_2(M); \mathbb{Z}) \cong \mathbb{Z}$ (UCT) rep. of generator (as a 2-cochain)

~~is ω : ...~~ . NB for any $\omega \in C^2$

$\delta\omega = 0$

is a cocycle (as no 3-simplices)

note: $\alpha \cup \beta + \beta \cup \alpha =$

: $V - W \longmapsto 0$ so $\alpha \cup \beta = -\beta \cup \alpha$.

ω : $\quad \alpha \cup \beta (V - W) = 1 \qquad \beta \cup \alpha (V - W) = -1$.

so cup product ring structure is

dim	0	1	2
	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
	1	α, β	ω

$\alpha \cup \beta = -\beta \cup \alpha = \omega \quad \alpha \cup \alpha = 0$

$\beta \cup \beta = 0$.