gives $\cdots \leftarrow H^{n+1}(A) \xleftarrow{\delta} H^n(A) \leftarrow H^n(X) \leftarrow H^n(X,A) \xleftarrow{\delta} H^{n-1}(A) \leftarrow \cdots$

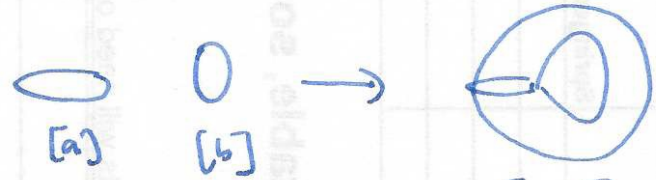
- induced homomorphisms $f: X \to Y$ gives $f^*: H^n(Y) \to H^n(X)$
- homotopy invariance: $f \simeq g: X \to Y$ gives $f^* = g^*: H^n(Y) \to H^n(X)$
- axioms for cohomology
- simplicial / singular / cellular cohomology
- Mayer-Vietoris sequence.

§3.2 Cup product

X CW-complex

$H^*(X;G)$ where $G = R$ a ring $[\mathbb{Z}, \mathbb{Q}, \mathbb{Z}/n, \dots]$

intuition: $H^k(X) \times H^\ell(X) \longrightarrow H^{k+\ell}(X \times X)$



$[a]$ $[b]$ $[a \times b]$

$X \times X$ has a cell structure with cells which are products of cells in X, X.

$$(\varphi \times \psi)(\sigma \times \tau) = \begin{cases} \varphi(\sigma)\psi(\tau) & \text{if } \dim(\sigma) = k,\ \dim(\tau) = \ell. \\ 0 & \text{else} \end{cases}$$

($\sigma \times \tau$: product cell)

consider: $\Delta: X \to X \times X$, $x \mapsto (x,x)$ define: $[\varphi] \cup [\psi] = \Delta^* [\varphi \times \psi]$

$$\begin{matrix} H^k(X) \times H^\ell(X) & \longrightarrow & H^{k+\ell}(X \times X) & \xrightarrow{\Delta} & H^{k+\ell}(X) \\ [\varphi] \quad [\psi] & & [\varphi \times \psi] & & \Delta^*[\varphi \times \psi]. \end{matrix}$$

Problem: Δ not a cellular map. Solutions (§3.B) • homotope Δ to a cellular map
• use singular homology

Fact $H^*(X;R) = \bigoplus_k H^k(X;R)$ becomes a ring w/ (cup) product

associative, distributive, not commutative: $\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha$

Reason: $\tau: X \times X \to X \times X$, $(x_1, x_2) \mapsto (x_2, x_1)$ does not act trivially on $H^*(X \times X)$. In odd dimensions

$1 \times a \ \boxed{a \times a} \xrightarrow{\tau} \boxed{a \times a}$ $\tau_*(a \times a) = -a \times a$.

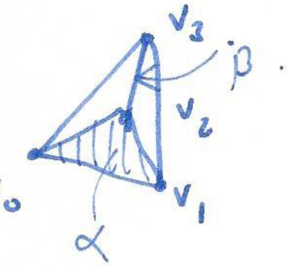
Defn (cup product at chain level) R ring $\mathbb{Z}, \mathbb{Z}/n, \mathbb{Q}\dots$

$$\begin{array}{ccccc} C^k(X;R) & \times & C^l(X;R) & \longrightarrow & C^{k+l}(X;R) \\ \alpha & & \beta & & \alpha\cup\beta \end{array}$$

by $$(\alpha\cup\beta)(\sigma) = \alpha(\sigma|_{[v_0,\dots,v_k]})\,\beta(\sigma|_{[v_k,\dots,v_n]})$$

where $\sigma: \Delta^{k+l} \to X$ is in $C^{k+l}_{k+l}(X;R)$

Example $k=2$, $l=1$



Lemma $\delta(\alpha\cup\beta) = \delta\alpha\cup\beta + (-1)^k\alpha\cup\delta\beta$.

Proof $$\delta(\alpha\cup\beta)(\sigma) = \alpha\cup\beta(\partial\sigma) = \alpha\cup\beta\Big(\sum_{i=0}^{k+l+1}(-1)^i\sigma|_{[v_0,\dots,\hat{v}_i,\dots v_{k+l+1}]}\Big)$$
$$= \sum_{i=0}^{k+l+1}(-1)^i\alpha(\sigma|_{[v_0,\dots\hat{v}_i\dots v_{k+l+1}]})\beta(\sigma|_{[\dots]}),$$

$$(\delta\alpha\cup\beta)(\sigma) = \delta\alpha(\sigma|_{[v_0,\dots v_{k+1}]})\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}).$$
$$= \alpha(\partial(\sigma|_{[v_0,\dots,v_{k+1}]}))\,\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}).$$
$$= \alpha\Big(\sum_{i=0}^{k+1}(-1)^i\sigma|_{[v_0,\dots,\hat{v}_i,\dots v_{k+1}]}\Big)\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$
$$= \sum_{i=0}^{k+1}\alpha(\sigma|_{[v_0,\dots,\hat{v}_i,\dots v_{k+1}]})\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}).$$

$$(-1)^k\alpha\cup\delta\beta = (-1)^k\alpha(\sigma|_{[v_0,\dots v_k]})\beta\Big(\sum_{i=k+1}^{k+l+1}(-1)^{i-k+1}\beta(\sigma|_{[v_{k+1},\dots\hat{v}_i,\dots v_{k+l+1}]})\Big)$$

so $\delta(\alpha\cup\beta) = \delta\alpha\cup\beta + (-1)^k\alpha\cup\delta\beta$ □.

Note · α, β cocycles, i.e. $\delta\alpha = 0$, $\delta\beta = 0$

then $\alpha \cup \beta$ is a cocycle, as $\delta(\alpha\cup\beta) = \underset{0}{\delta\alpha}\cup\beta + (-1)^k \alpha\cup\underset{0}{\delta\beta} = 0$.

· if α is also coboundary, i.e. $\alpha = \delta\phi$. then

$$\alpha\cup\beta = \delta\phi\cup\beta = \delta(\phi\cup\beta) = \delta\phi\cup\beta + (-1)^k \phi\cup\underbrace{\delta\beta}_{0}.$$

· if $\beta = \delta\phi$ then

$$\alpha\cup\beta = \alpha\cup\delta\phi = (-1)^k\delta(\alpha\cup\beta)$$

$\Rightarrow$ get induced map $\cup : H^k(X;R)\times H^l(X;R) \to H^{k+l}(X;R)$

$$\alpha\ ,\ \beta \longmapsto \alpha\cup\beta.$$

· associative, distributive, follows from chain level.

Remark Def^n: $H^*(X;R) = \bigoplus_n H^n(X;R)$ is a ring under cup product (not nec. commutative)

Furthermore, if R has a unit 1_R, then so does H^*:

$1_{H^*} \in H^0(X;R)$ given by $1_{H^*}(\text{0-simplex}) = 1_R$.

Example $\mathbb{C}P^2$: cellular chain:

$$\begin{array}{lccccccccccc} & & 4 & & 3 & & 2 & & 1 & & 0 & \\ & 0 \to & \mathbb{Z} & \to & 0 & \to & \mathbb{Z} & \to & 0 & \to & \mathbb{Z} & \to 0 \\ H_*: & & \mathbb{Z} & & 0 & & \mathbb{Z} & & 0 & & \mathbb{Z} & \\ H^*: & & \mathbb{Z} & & 0 & & \mathbb{Z} & & 0 & & \mathbb{Z}. & \\ & & \beta & & & & \alpha & & & & 1. & \end{array}$$

note $\alpha\cup\beta = 0 = \beta\cup\beta$

Q: $\alpha\cup\alpha$? fact $\alpha\cup\alpha = \beta$. (so $H^*(\mathbb{C}P^2) \cong \mathbb{Z}[\alpha]/\alpha^3$).

Example $S^2 \vee S^4$ same H^*, but $\alpha\cup\alpha = 0$.