recall $C_* = \{ \cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots \}$

$C^* = \mathrm{Hom}(C_*, G) = \{ \cdots \leftarrow C_{n+1}^* \xleftarrow{\delta} C_n^* \leftarrow C_{n-1}^* \leftarrow \cdots \}$

$\delta(\varphi) = \varphi \cdot \partial$.

H_n = homology of C_*

H^n = (co)homology of C^*

Universal coefficients Thm: The following is split exact:

$$0 \to \mathrm{Ext}(H_{n-1}, G) \to H^n(C_*; G) \xrightarrow{h} \mathrm{Hom}(H_n, G) \to 0$$

Ext: derived functor of $\mathrm{Hom}(\cdot, G)$

free resolution (exact, F_i free)

$$\to F_2 \to F_1 \to F_0 \to H \to 0$$

$$\leftarrow F_2^* \leftarrow F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0$$

$\mathrm{Ext}(H, G)$ = cohomology here.

previously: h is onto

Ext is well defined.

Proof (UCT) set $Z_n = \ker \partial_n$ $\quad B_n = \mathrm{im}\, \partial_{n+1}$

$$0 \to \ker h \to H^n(C_*; G) \xrightarrow{h} \mathrm{Hom}(H_n, G) \to 0$$

short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

} each row splits, as all groups are free.

Apply $\mathrm{Hom}(\cdot, G)$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & Z_{n+1}^* \arrow[l] \arrow[u] & C_{n+1}^* \arrow[l] \arrow[u] & B_n^* \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & Z_n^* \arrow[l] \arrow[u, "0"] & C_n^* \arrow[l, "i_n^*"'] \arrow[u, "\delta"] & B_{n-1}^* \arrow[l] \arrow[u, "0"] & 0 \arrow[l] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

$\phi_1 \circ \partial \xleftarrow{\delta} \phi_1|_{B_n} = \phi_0|_{B_n}$

$\phi_0 \longmapsfrom \phi_1$

$\Downarrow$

each row is exact

take long exact sequence:

$$\cdots \leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(C, G) \xleftarrow{\delta} B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \cdots$$

induced homomorphism is just the dual to the inclusion $B_n \xrightarrow[i_n]{} Z_n$.

split out short exact sequences:

so

$$0 \leftarrow \ker i_n^* \xleftarrow{h} H^n(C, G) \leftarrow \operatorname{coker} i_{n-1}^* \leftarrow 0$$

$\ker i_n^* = \mathrm{Hom}(H_n, G)$

~~claim~~ **want**: $\operatorname{coker} i_{n-1}^* = \mathrm{Ext}(H_{n-1}, G)$

note:

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C) \longrightarrow 0 \quad \text{short exact}$$

but this is a free resolution for $H_{n-1}(C)$!

dual:

$$0 \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow H_{n-1}(C)^* \leftarrow 0$$

cokernel of this is exactly 1st-cohomology group of free resolution, i.e. $\operatorname{coker} i_{n-1}^* = \mathrm{Ext}(H_{n-1}, G)$. $\square$.

Universal coefficients for homology

C_* chain complex of free abelian groups.

The following is split exact: $0 \to H_n(C)\otimes G \to H_n(C;G) \to \mathrm{Tor}(H_{n-1}(C),G) \to 0$

Recall tensor products for vector spaces: V basis $v_1,\dots v_m \in B_V$
W basis $w_1,\dots, w_n \in B_W$

$V\otimes W$ has basis $\{ v_i w_j \mid v_i \in B_V, w_j \in B_W\}$, i.e. $\dim(V\otimes W) = \dim(V)\dim(W)$.

tensor products for abelian groups: A,B abelian groups

$A\otimes B$ is an abelian group: generators $a\otimes b$ $a\in A$, $b\in B$

relations: $(a+a')\otimes b = a\otimes b + a'\otimes b$
$a\otimes(b+b') = a\otimes b + a\otimes b'$

identity: $\text{zero} = 0\otimes 0 = 0\otimes b = a\otimes 0$

Inverses: $-(a\otimes b) = (-a)\otimes b = a\otimes(-b)$

useful facts: $A\otimes B \cong B\otimes A$

$$(\oplus A_i)\otimes B \cong \oplus (A_i\otimes B)$$

$$(A\otimes B)\otimes C \cong A\otimes(B\otimes C)$$

$$\mathbb{Z}\otimes A \cong A$$

$$\mathbb{Z}_n\otimes A \cong A/n\mathbb{Z}A$$

- homomorphisms $f: A\to A'$, $g: B\to B'$ induce a homomorphism $f\otimes g: A\otimes B \to A'\otimes B'$
  $a\otimes b \mapsto f(a)\otimes g(b)$

- a bilinear map $\phi: A\times B \to C$ induces a homomorphism $A\otimes B \to C$
  $a\otimes b \mapsto \phi(a,b)$

How to compute Tor

1) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$
2) $\mathrm{Tor}(\bigoplus A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$
3) $\mathrm{Tor}(A,B) = 0$ if A (or B!) is torsion free
4) $\mathrm{Tor}(\mathbb{Z}/n, B) = \ker(B \xrightarrow{n} B)$

Example $\mathrm{Tor}(\underset{\mathbb{Z}_2 \oplus \mathbb{Z}_3}{\underset{\shortparallel}{\mathbb{Z}_6}}, \underset{\mathbb{Z}_3 \oplus \mathbb{Z}_5}{\underset{\shortparallel}{\mathbb{Z}_{15}}}) = \underset{0}{\mathrm{Tor}(\mathbb{Z}_2, \mathbb{Z}_3)} \oplus \underset{0}{\mathrm{Tor}(\mathbb{Z}_2, \mathbb{Z}_5)} \oplus \underset{\mathbb{Z}_3.}{\mathrm{Tor}(\mathbb{Z}_3, \mathbb{Z}_3)} \oplus \underset{0}{\mathrm{Tor}(\mathbb{Z}_3, \mathbb{Z}_5)}$

Def^n $\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \to 0$ free resolution for A.

$\otimes B$:

$$\cdots \to F_2 \otimes B \xrightarrow{f_2 \otimes 1} F_1 \otimes B \xrightarrow{f_1 \otimes 1} F_0 \otimes B \xrightarrow{f_0 \otimes 1} A \otimes B \to 0$$

not nec. exact exact exact

Def^n $H_1(F_A \otimes B) = \mathrm{Tor}(A,B)$.

Thm This is well defined $\square$.

Exercise: $H_n(C) = \mathbb{Z}^d \oplus \mathbb{Z}/p^{k_1} \oplus \mathbb{Z}/p^{k_2} \oplus \cdots \oplus \mathbb{Z}/p^{k_n} \oplus$ torsion coprime to p

$H_{n-1}(C) = \mathbb{Z}^e \oplus \mathbb{Z}/p^{l_1} \oplus \cdots \oplus \mathbb{Z}/p^{l_m} \oplus$ torsion coprime to p

then $H_n(C; \mathbb{Z}/p) \cong (\mathbb{Z}/p)^{d+n+m} \cong H^n(C; \mathbb{Z}/p)$.

§ Cohomology of spaces

X space, G abelian group

$C^n(X;G) = \mathrm{Hom}(C_n(X), G)$ singular cochains with coefficients in G

note: $H^0(X;G) \cong \mathrm{Hom}(H_0(X), G)$

$H^1(X;G) \cong \mathrm{Hom}(H_1(X), G) \cong \mathrm{Hom}(\pi_1 X, G)$

also get:
- reduced cohomology groups $\tilde{H}^n(X;G)$
- relative cohomology groups $H^n(X,A;G)$
- long exact sequence of the pair