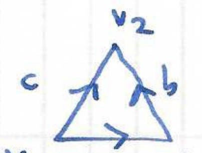
<u>intuition</u> $H^0(X;G) = \ker(\delta) < \Delta^0(X;G)$

$\delta\phi = 0 \iff \phi(v_0) = \phi(v_1)$ for all edges $[v_0, v_1]$

i.e. $H^0(X;G) = G^{\#\text{connected components of } X}$

$H^1(X;G) = \ker(\delta)/\operatorname{im}(\delta)$ $\qquad \psi \in \Delta^1(X;G) \qquad \psi : \text{edges} \to G$

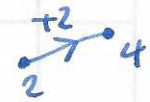
$\delta\psi = 0 \iff$ "locally trivial"



"going around a loop which bounds gives 0"

$\delta\psi(\sigma) = \psi(\partial\sigma) = a + b - c$

$\operatorname{im}(\delta) =$ edge functions arising from functions on vertices



<u>e.g.</u> S^1

$\leftarrow$ doesn't arise from $\phi : v_i \mapsto \mathbb{Z}$.

<u>§3.1 Cohomology groups</u>

<u>Universal coefficients theorem</u> (UCT). The following short ~~exact~~ sequence

$$0 \to \operatorname{Ext}(H_{n-1}, G) \to H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n, G) \to 0$$

is split ~~exact~~, i.e. $H^n(C;G) \cong \operatorname{Hom}(H_n, G) \oplus \operatorname{Ext}(H_{n-1}, G)$.

<u>Ext</u>

<u>Defn</u> A free resolution of an abelian group H is an exact sequence

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to H \to 0$$

where each F_i is free abelian.

<u>Examples</u>

$$\cdots \to 0 \to 0 \to \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

$$\cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

Remark $\mathrm{Hom}(\cdot, G)$ does not preserve exact sequences.

Example

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

$\mathrm{Hom}(\cdot, \mathbb{Z})$:

$$0 \leftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \leftarrow 0 \leftarrow 0$$

H^*: $\quad 0 \qquad \mathbb{Z}/2 \qquad 0 \qquad 0 \qquad 0$

Defn $\mathrm{Ext}(H, G)$ is the first cohomology group of

$$\cdots \leftarrow F_2^* \leftarrow F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0$$

where $A^* = \mathrm{Hom}(A, G)$ and F_i is any free resolution of H.

Remark every abelian group has free resolution $\to 0 \to 0 \to R \to L \to H \to 0$ so all other cohomology groups zero.

Properties

- $\mathrm{Ext}(H \oplus H', G) = \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$
- H free $\Rightarrow$ $\mathrm{Ext}(H, G) = 0$ $\quad \left[0 \to 0 \to H \xrightarrow{\cong} H \to 0 \right]$.

$$0 \leftarrow 0 \leftarrow H^* \xleftarrow{\cong} H^* \leftarrow 0$$

- $\mathrm{Ext}(\mathbb{Z}/n, G) = G/nG$

$$\left[0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n \to 0 \right]$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ G & G & G. \end{array}$$

$$0 \leftarrow \mathbb{Z}^* \xleftarrow{\times n} \mathbb{Z}^* \leftarrow \mathbb{Z}/n^* \leftarrow 0$$

$$\begin{array}{ccc} \mathbb{Z}^* & & \mathbb{Z}^* \\ \cong & & \cong \\ G & \xleftarrow{\times n} & G \end{array}$$

Examples

$$\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}) \cong \underbrace{\mathrm{Ext}(\mathbb{Z}, \mathbb{Z})}_{\cong\, 0} \oplus \underbrace{\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z})}_{\cong\, \mathbb{Z}/2} \cong \mathbb{Z}/2$$

$$\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2) \cong \underbrace{\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/2)}_{\cong\, 0} \oplus \underbrace{\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/2)}_{\cong\, \mathbb{Z}/2} \cong \mathbb{Z}/2$$

$$\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/3) \cong \underbrace{\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/3)}_{\cong\, 0} \oplus \underbrace{\mathrm{Ext}(\mathbb{Z}/2, \mathbb{Z}/3)}_{\cong\, 0} \cong 0$$

<u>claim</u> Ext well defined.

<u>proof</u> (sketch)

<u>Lemma</u> Let H, H' be abelian groups w/ free resolutions F, F'.
Then any homomorphism $\alpha : H \to H'$ extends to a chain map

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r] \arrow[d, "\alpha_3"] & F_1 \arrow[r] \arrow[d, "\alpha_2"] & F_0 \arrow[r] \arrow[d, "\alpha_1"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\ \cdots \arrow[r] & F_2' \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & H' \arrow[r] & 0 \end{tikzcd}$$

and any two such extensions are chain homotopic.

<u>Proof</u> (sketch) (diagram chase)

define α_i:

$$g_1 \mapsto \partial g_1, \quad g_0 \text{ generator} \mapsto \partial g$$

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, "\alpha_1"] & F_0 \arrow[r] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\ F_1' \arrow[r] & F_0' \arrow[r] & H' \arrow[r] & 0 \end{tikzcd}$$

$$\alpha_1(g_1) \mapsto \alpha_0 \partial g_1 \qquad \alpha_1(g) \mapsto \alpha \partial g$$

define h: $\swarrow \alpha_i \downarrow \beta_i \swarrow h$ want $\partial h + h\partial = \alpha_i - \beta_i$

suffices to show $\alpha_i - \beta_i$ chain homotopic to 0, so suffices to consider

define h:

$$g_0 \text{ generator} \mapsto \partial g_0, \quad g \text{ generator}$$

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] \arrow[d, "\alpha_1"] \arrow[d, shift left=3, "0"'] & F_0 \arrow[r] \arrow[d, "\alpha_0"] \arrow[d, shift left=3, "0"'] \arrow[dl, "h"'] & H \arrow[r] \arrow[d, "0"] \arrow[dl, "h=0"'] & 0 \arrow[dl, "h=0"'] \\ \cdots \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & H' \arrow[r] & 0 \end{tikzcd}$$

$$\alpha_1(g_0) \quad \alpha_0(g_0) \mapsto 0 \qquad 0$$

check $\partial h + h\partial = \alpha_i$ $\square$.

<u>Lemma</u> for any two free resolutions F, F' of H, there are canonical isomorphisms $H^n(F;G) \cong H^n(F';G)$

<u>Corollary</u> $\mathrm{Ext}(H, G) \cong H^1(F;G)$ is well defined.

<u>Proof</u> (sketch)

$$\begin{tikzcd} F \arrow[r] \arrow[d, shift left, "\alpha"] & H \arrow[d, shift left, "1_H"] \\ F' \arrow[r] \arrow[u, shift left, "\beta"] & H \arrow[u, shift left] \end{tikzcd}$$

$\beta\alpha$ is chain homotopic to 1_F.

$$\begin{tikzcd} H^n(F;G) \arrow[d, shift right, "\beta^*"'] \\ H^n(F';G) \arrow[u, shift right, "\alpha^*"'] \end{tikzcd}$$

$\alpha^* \beta^* = 1_{H^n(F;G)}$

$\beta^* \alpha^* = 1_{H^n(F';G)}$.

so α^*, β^* isomorphisms; canonical by 1st Lemma $\square$.

<u>Thm</u> (UCT) $0 \to \mathrm{Ext}(H_{n-1}, G) \to H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n, G) \to 0$

is split exact.

<u>Proof</u> (sketch)

define h: consider $[\phi] \in H^n(C; G)$, $\phi \in C^n = \mathrm{Hom}(C_n, G)$ $\delta\phi = 0$

$[\alpha] \in H_n$, $\alpha \in C_n$

define: $h([\phi])([\alpha]) = \phi(\alpha) \in G$. (i.e. $h([\phi])(\cdot): H_n \to G$).

<u>check well defined</u>:

① replace α with $\alpha + \partial\beta$

$\phi(\alpha + \partial\beta) = \phi(\alpha) + \phi(\partial\beta) = \phi(\alpha) + \underbrace{\delta\phi}_{0}(\beta) = \phi(\alpha)$ ✓.

② replace ϕ with $\phi + \delta\psi$

$(\phi + \delta\psi)(\alpha) = \phi(\alpha) + \delta\psi(\alpha) = \phi(\alpha) + \psi(\underset{\overset{\shortparallel}{0}}{\partial\alpha}) = \phi(\alpha)$ ✓. □.

<u>Lemma</u> h is onto

<u>Proof</u> pick $\phi_0 \in \mathrm{Hom}(H_n, G)$ set $B_n = \mathrm{image}(\partial_n)$ $Z_n = \ker(\partial_n)$

consider:

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, shift left, "i"] & C_n \arrow[r, "\partial_n"] \arrow[l, shift left, "\rho"] \arrow[d, phantom, "\cong" rotate=-90] & B_{n-1} \arrow[r] & 0 \\ & & Z_n \oplus B_{n-1} & & \end{tikzcd}$$

short exact sequence of <u>free</u> abelian groups.

$\Rightarrow$ split

gives $\rho: C_n \to Z_n$ st. $\rho \circ i = \mathrm{id}|_{Z_n}$.

<u>define</u>: $\underset{\overset{\cap}{C^n(C;G)}}{\phi_1}: C_n \to G$ by

$$\begin{tikzcd} C_n \arrow[r, "\rho"] \arrow[rrr, bend right, "\phi_1"'] & Z_n \arrow[r, "q"] & H_n \arrow[r, "\phi_0"] & G \end{tikzcd}.$$

<u>check</u> cocycle: $\delta\phi_1(\underset{\in C_{n+1}}{\alpha}) = \phi_1(\partial\alpha) = \phi_0([\partial\alpha]) = 0$

by defn $h([\phi_1])([\alpha]) = \phi_1(\alpha) = \phi_0(\alpha)$, i.e. $h(\phi_1) = \phi_0$, so h is onto. □.