## Categories and functors

A category $\mathcal{C}$ consists of

1) a collection $ob(\mathcal{C})$ of objects.
2) sets $Mor(X,Y)$ of morphisms for each pair $X,Y \in Ob(\mathcal{C})$ including an "identity" morphism $\mathbb{1}_X \in Mor(X,X)$
3) a composition of morphisms function $\circ : Mor(X,Y) \times Mor(Y,Z) \to Mor(X,Z)$

s.t. for each $X,Y,Z \in Ob(\mathcal{C})$ $f \circ \mathbb{1} = f$ $\mathbb{1} \circ f = f$ and $(f \circ g) \circ h = f \circ (g \circ h)$.

**Examples**

| objects | morphisms |
|---|---|
| topological spaces | continuous maps |
| sets | functions |
| groups | homomorphisms |
| chain complexes | chain maps |
| long exact sequences | chain maps |

**Def^n** A functor $F : \mathcal{C} \to \mathcal{D}$ (covariant)

$$ob: X \longmapsto F(X)$$

$$Mor: \quad f \in M(X,Y) \longmapsto F(f) \in M(F(X), F(Y)) \quad \text{s.t. } F(\mathbb{1}) = \mathbb{1}$$

$$F(f \circ g) = F(f) \circ F(g)$$

**Examples** $\pi_1 : (Top, x_0) \to Groups$

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

$H_n$: $Top \longrightarrow$ Abelian groups

$$X \longmapsto H_n(X)$$

pairs $\longrightarrow$ exact sequences

$(X,A) \longmapsto$ long exact sequence of a pair.

**Def^n** A functor is contravariant if $F(f \circ g) = F(g) \circ F(f)$

<u>Examples</u> vector spaces $\longrightarrow$ vector spaces

$V \longmapsto V^{dual}$

Top $\longrightarrow$ abelian groups

$X \longmapsto H^n(X)$ cohomology.

<u>Defn</u> Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A <u>natural transformation</u> $T$ from $F$ to $G$ assigns a morphism $T_X : F(X) \to G(X)$ for each object $X$ in $\mathcal{C}$, s.t. for each $f \in M(X,Y)$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"] & F(Y) \arrow[d, "T_Y"] \\ G(Y) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

commutes.

<u>Example</u> The boundary map in the long exact sequence of a pair is natural

$f : (X,A) \to (Y,B)$ gives ~~$H_n(X,A) \xrightarrow{\partial} H_{n-1}(A$~~

$$\begin{tikzcd} H_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial"] & H_n(Y,B) \arrow[d, "\partial"] \\ H_{n-1}(A) \arrow[r, "f_*"] & H_{n-1}(B) \end{tikzcd}$$

commutes.

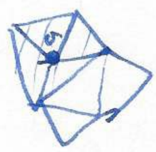
## <u>§2.C. Simplicial approximation</u>

<u>Defn</u> If $K, L$ are simplicial complexes, then a map $f : K \to L$ is <u>simplicial</u> if it sends each simplex of $K$ to a simplex of $L$ by a linear map, taking vertices to vertices.

<u>Thm</u> If $K$ is a finite simplicial complex, and $L$ is an arbitrary simplicial complex, then any map $f : K \to L$ is homotopic to a map which is simplicial with respect to some iterated barycentric subdivision of $K$.

<u>Observation</u> may have to subdivide, consider $f : S^1 \to S^1$ 

**Def$^{\underline{n}}$** Let $\sigma$ be a simplex in a simplicial complex $X$. The *star* of $\sigma$, $St(\sigma)$, is the union of all simplices containing $\sigma$. The *open star* $st(\sigma)$ is the union of all interiors of simplices containing $\sigma$.

**Example**  $st(\sigma)$ open and $\overline{st(\sigma)} = St(\sigma)$.

**Proof** choose metric on $K$ s.t. each simplex isometric to standard simplex in $\mathbb{R}^{n+1}$. $\{st(v) \mid v \in L^0\}$ is an open cover of $L$, so $f^{-1}(st(\sigma))$ is an open cover of $K$, with Lebesgue number $\epsilon$. Subdivide $K$ until diam of each simplex $< \epsilon/2$, call new complex $K$.

**Example**  

for each $v \in K^0$ choose $w \in L^0$ s.t. $st(v) \subset f^{-1}(st(w))$. This defines $g: K^0 \to L^0$, extend to $g: K \to L$ linearly on simplices.

**claim**: $f \simeq g$: use linear homotopy as each $f(x), g(x)$ contained in a common simplex, as $st(v_1) \cap \dots \cap st(v_n) = \phi$, unless $v_i$ are vertices of a common simplex $\sigma$, in which case $st(v_1) \cap st(v_2) \cap \dots \cap st(v_n) = \sigma$. $\square$.

**Observation**: if you subdivide $L$, can make homotopy arbitrarily small.

---

## §3 Cohomology

**so far**

$\pi_1 X$ = homotopy classes of maps $S^1 \to X$

Homology: $H_k(X; G)$ ($k$-cycles/boundaries).

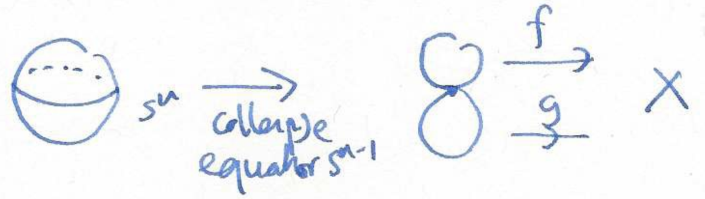
**next**

$\pi_n X$ = homotopy classes of maps $S^n \to X$.

cohomology $H^k(X; G)$ (algebraic dual to homology).

Higher homotopy groups $\pi_n(X, x_0) =$ homotopy classes $f$: $(S^n, s_0) \to (X, x_0)$.

composition: $[f] * [g] =$




Fact: this is abelian!

Example

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\pi_n(S^1)$ | $\mathbb{Z}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\pi_n(S^2)$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/12$ | $\mathbb{Z}/2$ |

Good sees homotopy type of $X$

Thm [Whitehead] $f: X \to Y$ map of CW complexes s.t. $f_*: \pi_n X \xrightarrow{\cong} \pi_n Y$ iso for all $n$.

Then $X \simeq Y$ homotopy equivalent.

Bad hard to compute: don't know $\pi_n(S^2)$! for all $n$....

(excision fails).

---

Cohomology vs homology

$$\mathrm{Top} \longrightarrow \text{Abelian Groups}$$

Hom homology: $X \xrightarrow{f} Y$ $\qquad H_k(X) \xrightarrow{f_*} H_k(Y)$ (covariant).

Cohom cohomology: $X \xrightarrow{f} Y$ $\qquad H^k(X) \xleftarrow{f^*} H^k(Y)$ (contravariant)

similarities $H^k$ defined by chain complex
long exact sequence of a pair
excision, MV sequence...

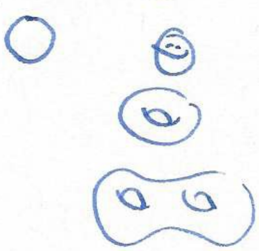
Hom, Cohom determine each other; for a field $F$, $H_k(X;F) \cong H^k(X;F)$

extra fact: $H^*(X)$ has a multiplication

$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X) \quad [\text{cup product}]$$

special case $X =$ manifold, space locally homeomorphic to $\mathbb{R}^n$

1 2 3 4



$S^3$ $\quad S^4$ $\quad \dots$

fact: on smooth manifolds, can do calculus, $H^k(M;\mathbb{R})$ can be defined in terms of differential forms, e.g. $\omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$ is a generator for $H^2(S^2;\mathbb{R}) \cong \mathbb{R}$.

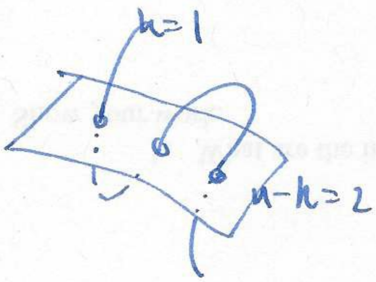
Poincaré duality $M$ closed compact $n$-manifold, then $H_k(M;\mathbb{Z}/2\mathbb{Z}) \cong H^{n-k}_{(n-k)}(M;\mathbb{Z}/2)$

If $M$ orientable, then $H_k(M;G) \cong H^{n-k}(M;G)$.

comes from: $H_k(M;\mathbb{Z}/2) \times H_{n-k}(M;\mathbb{Z}/2) \to \mathbb{Z}/2$ intersection form mod 2.

PD $\iff$ this is non-degenerate

$n=3$




intersection form dual to cup product

$H^{n-k}(M) \times H^k(M) \to H^n(M) \cong \mathbb{Z}/2$.

---

$X$ topological space

Homology: $\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$

$C_n = C_n(X;\mathbb{Z}) = \bigoplus_{\sigma} \mathbb{Z}$ (singular/cellular) chain groups.

$\sigma$ ↖ simplex

cochains: $C^n(X;G) = \mathrm{Hom}(C_n, G) = \prod_{\sigma} G$

$=$ functions from the set of simplices to $G$.

$\xleftarrow{\delta_{n+1}} C^{n+1} \xleftarrow{\delta_n} C^n \xleftarrow{\delta_{n-1}} C^{n-1} \xleftarrow{\delta}$

coboundary map: $\phi: C_n \to G$ set $\delta_n(\phi) = \phi \circ \partial_{n+1}$

$\in C^n$