§2.3 Formal view

Axioms for homology

A (reduced) homology theory assigns to each (non-empty) CW-complex X a sequence of abelian groups $\tilde{h}_n(X)$, and to each map $f: X \to Y$ a homomorphism $f_*: \tilde{h}_n(X) \to \tilde{h}_n(Y)$ s.t. $(fg)_* = f_* g_*$ and $\mathbb{1}_* = \mathbb{1}$,

such that:

1) $f \simeq g : X \to Y$ then $f_* = g_* : \tilde{h}_n(X) \to \tilde{h}_n(Y)$

2) there are boundary maps $\partial : \tilde{h}_n(X/A) \to \tilde{h}_{n-1}(A)$

defined for each CW pair (X, A), with an exact sequence

$$\cdots \xrightarrow{\partial} \tilde{h}_n(A) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} h_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A)$$

which are <u>natural</u>, i.e. given $f: (X,A) \to (Y,B)$

with quotient map $\bar{f} : X/A \to Y/B$, then

$$\begin{tikzcd} h_n(X/A) \arrow[r, "\partial"] \arrow[d, "f_*"] & h_{n-1}(A) \arrow[d, "f_*"] \\ h_n(Y/B) \arrow[r, "\partial"] & h_{n-1}(B) \end{tikzcd}$$

commutes.

3) if $X = \bigvee_\alpha X_\alpha$, $i_\alpha : X_\alpha \hookrightarrow X$, then $\oplus i_\alpha : \bigoplus_\alpha \tilde{h}_n(X_\alpha) \xrightarrow{\cong} \tilde{h}_n(X)$

Isomorphism.

Examples $\tilde{H}_n(X)$ reduced singular homology $\}$ also w/ coeffs.

$\tilde{H}_n^{CW}(X)$ reduced cellular homology