Recall Thm [Mayer-Vietoris] $X = \mathrm{int}(A) \cup \mathrm{int}(B)$ gives long exact sequence: (25)

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

recall: subdivision, set $\mathcal{U} = \{\mathrm{int}\, A, \mathrm{int}\, B\}$

$C_n^{\mathcal{U}}(X) =$ chains that are sums of chains in A and chains in B.

∂ preserves this property, so $(C_n^{\mathcal{U}}(X), \partial)$ is a chain complex.

claim:

$$0 \to C_n(A \cap B) \xrightarrow[\text{"}i - j\text{"}]{\phi} C_n(A) \oplus C_n(B) \xrightarrow[\text{"}i_A + i_B\text{"}]{\psi} C_n^{\mathcal{U}}(X) \to 0$$

$$x \longmapsto (x, -x)$$ is exact.

$$(x, y) \longmapsto x + y$$

$$\begin{tikzcd} & A \arrow[dr, hook, "i_A"] & \\ A \cap B \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & X \\ & B \arrow[ur, hook, "i_B"'] & \end{tikzcd}$$

Proof (of claim)

- $\ker \phi = 0$ ✓ $C_n(A \cap B) \subset C_n(A)$
- $\psi\phi(x) = \psi(x - x) = x - x = 0$ so $\mathrm{im}\, \phi \subset \ker \psi$
- $\ker \psi \subset \mathrm{im}\, \phi$: s'pose $(x, y) \in C_n(A) \oplus C_n(B)$

  and $x - y = 0$ in $C_n^{\mathcal{U}}(X) \Rightarrow x = y$ in $C_n^{\mathcal{U}}(X)$

  $\Rightarrow x, y$ are chains in $C_n^{\mathcal{U}}(A \cap B) = C_n(A \cap B)$

  so lie in image of ϕ.
- ψ surjective by defn of $C_n^{\mathcal{U}}(X)$ $\square$

Proof (of MV Thm) short exact sequence of chain complexes gives long exact sequence of homology groups. $\square$

boundary map : $H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B)$

pick $\alpha \in H_n(X)$ represented by cycle z, subdivide so $z = x + y$

$x \in C_n(A)$, $y \in C_n(B)$ (not nec. cycles!)

but $\partial z = 0 = \partial x + \partial y$, i.e. $\partial x = -\partial y$, but this implies $\partial x = -\partial y \in C_{n-1}(A \cap B)$

so map $\alpha \longmapsto \partial\alpha = [\partial x] = [-\partial y]$.

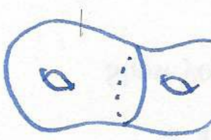
Example ①

$A \simeq S^1 \vee S^1$ B. $A \cap B = \bigcirc \simeq S^1$

$H_k($$)$ $= \mathbb{Z}$ $k=0$
$= \mathbb{Z}^2$ $k=1$.
$= 0$ else.

$$\begin{array}{l}
\overset{0}{H_2(A\cap B)} \to \overset{0}{H_2(A)} \oplus \overset{0}{H_2(B)} \to H_2(X) \Rightarrow \mathbb{Z} \\
\to \overset{\mathbb{Z}}{H_1(A\cap B)} \xrightarrow{0} \overset{\mathbb{Z}^2}{H_1(A)} \oplus \overset{0}{H_1(B)} \to H_1(X) \Rightarrow \mathbb{Z}^2 \to 0 \\
\qquad 1 \longmapsto (0,0). \\
\to \overset{\mathbb{Z}}{H_0(A\cap B)} \to \overset{\mathbb{Z}}{H_0(A)} \oplus \overset{\mathbb{Z}}{H_0(B)} \to \overset{\mathbb{Z}}{H_0(X)} \\
\qquad x \longmapsto (x, -x)
\end{array}$$

②



A B.

$A \cap B \simeq S^1$

$$\begin{array}{l}
\overset{0}{H_2(A\cap B)} \to \overset{0}{H_2(A)} \oplus \overset{0}{H_2(B)} \to \overset{\mathbb{Z}}{H_2(X)} \\
\to \overset{\mathbb{Z}}{H_1(A\cap B)} \to \overset{\mathbb{Z}^2}{H_1(A)} \oplus \overset{\mathbb{Z}^2}{H_1(B)} \to \overset{\mathbb{Z}^4}{H_1(X)} \to 0 \\
\qquad 1 \longmapsto (0,0) \\
\to \overset{\mathbb{Z}}{H_0(A\cap B)} \to \overset{\mathbb{Z}}{H_1(A)} \oplus \overset{\mathbb{Z}}{H_1(B)} \to H_0(X) \\
\qquad 1 \longmapsto (1,-1)
\end{array}$$

Thm (Brouwer) invariance of domain ($\mathbb{R}^n \not\cong \mathbb{R}^m$ if $m \neq n$). (27)

If non-empty open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$.

<u>Proof</u> pick $x \in U$ and consider $H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ by excision. Long exact sequence of a pair:

$$\cdots \to H_n(\mathbb{R}^m \setminus \{x\}) \to H_n(\mathbb{R}^m) \to H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \to \cdots$$

$$\Rightarrow \quad H_n(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong H_{n-1}(\mathbb{R}^m \setminus \{x\})$$

↑ deformation retracts to S^{m-1}

$$\Rightarrow \quad H_k(U, U \setminus \{x\}) = \begin{cases} \mathbb{Z} & \text{if } k = m \\ 0 & \text{otherwise.} \end{cases}$$

a homeo $h : U \to V$ induces isomorphisms $H_k(U, U \setminus \{x\}) \to H_k(V, V \setminus \{h(x)\})$

$\Rightarrow m = n$. $\square$.

<u>**Defn**</u> The homology groups $H_n(X, X \setminus \{x\})$ are the <u>local homology groups</u> of X at x

<u>Examples</u>: find explicit cycles representing generators of $H_n(D^n, \partial D^n)$ and $H_n(S^n)$

<u>note</u>: $(D^n, \partial D^n) \cong (\Delta^n, \partial \Delta^n)$

<u>claim</u>: $i_n : \Delta^n \to \Delta^n$ is a generator for $H_n(\Delta^n, \partial \Delta^n) \cong \mathbb{Z}$.

<u>$n=0$</u>: ✓

<u>induction step</u>: Let $\Lambda \subset \Delta^n$ be the union of all but one $(n-1)$-dim faces of Δ^n.

Consider $$H_n(\Delta^n, \partial \Delta^n) \xrightarrow[\text{①}]{\cong} H_{n-1}(\partial \Delta^n, \Lambda) \xleftarrow[\text{②}]{\cong} H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1}).$$

① long exact sequence of the triple $\Lambda \subset \partial \Delta^n \subset \Delta^n$

recall : $0 \to C_k(\partial\Delta^n, \Lambda) \to C_k(\Delta^n, \Lambda) \to C_k(\Delta^n, \partial\Delta^n) \to 0$ exact

gives :

$$\cdots \to H_k(\partial\Delta^n, \Lambda) \to H_k(\Delta^n, \Lambda) \to H_k(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{k-1}(\partial\Delta^n, \Lambda) \to \cdots$$

Δ^n deformation retracts to Λ, so $H_k(\Delta^n, \Lambda) = 0$ for all k.

② consider $\Delta^{n-1} \subset \partial\Delta^n$ as missing face.

$(\Delta^{n-1}, \partial\Delta^{n-1}) \hookrightarrow (\partial\Delta^n, \Lambda)$ [for good pairs, $H_n(X,A) \cong H_n(X/A, A/A)$]

$$\begin{array}{ccccc} (\Delta^{n-1}, \partial\Delta^{n-1}) & \hookrightarrow & (\partial\Delta^n, \Lambda) & & \\ \downarrow q & & \downarrow q & & \\ \Delta^{n-1}/\partial\Delta^{n-1} & \underset{\text{homeo}}{\cong} & \partial\Delta^n/\Lambda & \underset{\text{homeo}}{\cong} & S^{n-1} \end{array}$$

therefore a generator ~~of~~ $i_n \in H_n(\Delta^n, \partial\Delta^n)$ is sent to ∂i_n, a generator in $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$, so $\partial i_n = \pm i_{n-1} \in C_{n-1}(\partial\Delta^n, \Lambda)$

recall if $S^{n-1} \underset{\substack{1pt \\ \text{homeo}}}{\cong} (\Delta^{n-1}, \partial\Delta^{n-1})$ then generated by identity map $\mathbb{1} : \Delta^{n-1} \to \Delta^{n-1}$ □.

Homology with coefficients

$\mathbb{Z}$-coefficients : consider chains $\sum n_i \sigma_i$ $n_i \in \mathbb{Z}$

G-coefficients : $\sum n_i \sigma_i$ $n_i \in G$ abelian group

still get a chain complex

$$\cdots \to C_n(X; G) \xrightarrow{\partial} C_{n-1}(X; G) \to \cdots$$

homology groups : $H_n(X; G)$ "homology of X with coefficients in G".

Example $X =$ Moore space $M(\mathbb{Z}/m\mathbb{Z}, 1)$ = attach a 2-cell e^2 to S^1 by a map of degree m.

Cellular homology chain: $0 \to \overset{2}{\mathbb{Z}} \xrightarrow{d} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \to 0$

	2	1	0
$H_*(X;\mathbb{Z})$	0	$\mathbb{Z}/m\mathbb{Z}$	$\mathbb{Z}$
$H_*(X;\mathbb{Z}/m\mathbb{Z})$	$\mathbb{Z}/m\mathbb{Z}$	$\mathbb{Z}/m\mathbb{Z}$	$\mathbb{Z}/m\mathbb{Z}$

consider quotient map $X \xrightarrow{f} X/S^1 \cong S^2$

induces $f_* : H_k(X) \to H_k(S^2)$

<u>note</u>: $f_* = 0$ on $H_k(X;\mathbb{Z})$ for each k

<u>Q</u>: is f homotopic to a constant map?

<u>A</u>: no, consider long exact sequence of a pair (X, S^1) with $\mathbb{Z}/m\mathbb{Z}$ coeffs

$$
\begin{array}{ccccccccc}
0 \to H_2(S^1;\mathbb{Z}/m\mathbb{Z}) & \to & H_2(X;\mathbb{Z}_m) & \xrightarrow{f_*} & H_2(X/S^1;\mathbb{Z}_m) & \to & H_1(S^1;\mathbb{Z}_m) \to \cdots \\
0 & \to & \mathbb{Z}_m & \to & \mathbb{Z}_m & & \mathbb{Z}_m
\end{array}
$$

$\uparrow$ must be injective

$\Rightarrow f \not\simeq$ constant map.