Mayer–Vietoris sequence

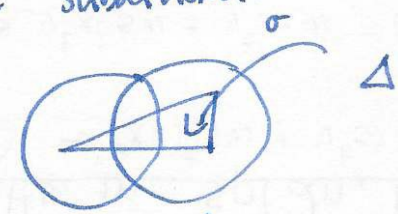
Thm. $X = A \cup B$, in fact $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, then get long exact sequence:

$$\cdots \to H_n(A\cap B) \xrightarrow{\text{"}i - j\text{"}} H_n(A) \oplus H_n(B) \xrightarrow{\text{"}i_A + i_B\text{"}} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

motivation if you know $H_k(A\cap B)$, $H_k(A)$, $H_k(B)$, can often work out $H_n(X)$.

proof: needs barycentric subdivision.

basic problems $X = A \cup B$

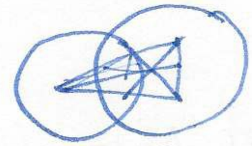


A B.

let $\sigma : \Delta \to X$ be a singular simplex

want to express σ as a sum of simplices in A, or in B.

solution (idea) barycentric subdivision:



A B

Def$\underline{\text{n}}$ an open cover is a collection of open sets $\mathcal{U} = \{U_i\}$ s.t. $X = \bigcup_i \mathrm{int}(U_i)$

define: $C_n^{\mathcal{U}}(X) \subset C_n(X)$

$\uparrow$

subgroup consisting of chains $\sum n_i \sigma_i$, s.t. $\sigma_i(\Delta) \subset U_j$ for some j.

note:

$$\begin{array}{ccc} C_n(U) & \xrightarrow{\partial} & C_{n-1}(U) \\ \cup & & \cup \\ C_n^{\mathcal{U}}(U) & \xrightarrow{\partial} & C_{n-1}^{\mathcal{U}}(U) \end{array}$$

so $C_n^{\mathcal{U}}(X)$ forms a chain complex.

Prop$\underline{\text{n}}$ The inclusion $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence, i.e. there is a chain map $p : C_n(X) \to C_n^{\mathcal{U}}(X)$ s.t. ip and pi are chain homotopic to the identity map.

Therefore i induces an isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$ for all n.

<u>recall</u>:

<u>Defn</u> We say a chain map $P: C_n(X) \to C_{n+1}(Y)$ is a <u>chain homotopy</u> between two chain maps $f_\#, g_\# : C_n(X) \to C_n(Y)$ if

$$\partial P + P\partial = g_\# - f_\#.$$

<u>Propn</u> chain homotopic chain maps induce the <u>same</u> homomorphisms on homology

<u>Proof</u>

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "P"'] & C_n(X) \arrow[r] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[r] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "P"'] & \cdots \arrow[dl] \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

Let $\alpha \in C_n(X)$ be a cycle, i.e. $\partial\alpha = 0$

then $f_\#(\alpha) - g_\#(\alpha) = \partial P(\alpha) + P\underbrace{\partial(\alpha)}_{=0} = \partial P(\alpha)$

↑ this is a boundary!

so $[f_\#(\alpha)] = [g_\#(\alpha)]$ in $H_n(Y)$! . $\square$.

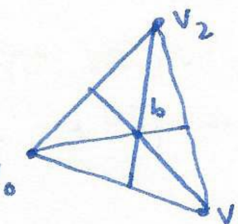
<u>setup</u>:

$$\begin{tikzcd} \cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, shift right, "\rho"'] & C_{n-1}(X) \arrow[r] \arrow[d, shift right, "\rho"'] & \cdots \\ \cdots \arrow[r] & C_n^{\mathcal{U}}(X) \arrow[r, "\partial"] \arrow[u, shift right, "i"'] & C_{n-1}^{\mathcal{U}}(X) \arrow[r] \arrow[u, shift right, "i"'] & \cdots \end{tikzcd}$$

want: F s.t. $\partial F + F\partial = i\rho - \mathbb{1}$

G s.t. $\partial G + G\partial = \rho i - \mathbb{1}$.

<u>Proof</u> 1 barycentric subdivision of simplices



a simplex $\sigma = [v_0, v_1, \dots, v_n]$ has barycentric coordinates $\sum_{i=0}^{n} t_i v_i$, $\sum t_i = 1$, $t_i \geq 0$

<u>Defn</u> The <u>barycenter</u> of σ is $b = \sum_{i=0}^{n} \frac{1}{n+1} v_i$

we can decompose σ into simplices $[b, w_0, w_1, \dots, w_{n-1}]$ where each $[w_0, \dots, w_{n-1}]$ is a face of the barycentric subdivision of of a face $[v_0, v_1, \dots \hat{v}_i, \dots v_n]$ of σ.

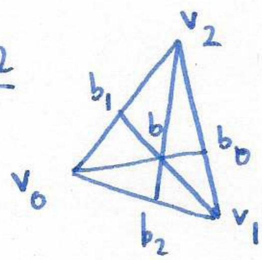
$\underline{n=0}$ $\quad [v_0] \rightsquigarrow [v_0]$

$\underline{n=1}$ $\quad [v_0, v_1]$

$b = \frac{1}{2}v_0 + \frac{1}{2}v_1$ $\quad$ faces of $[v_0, v_1]$ are $[v_0]$
$[v_1]$

so subsimplices are $[b, v_0]$ and $[b, v_1]$

$\underline{n=2}$



get $[b, b_0, v_1]$ $\quad [b, b_1, v_0]$ $\quad [b, b_2, v_0]$

$[b, b_0, v_2]$ $\quad [b, b_1, v_2]$ $\quad [b, b_2, v_2]$

$\underline{\text{claim}}$: diameter of each subsimplex of Δ^n at most $\frac{n}{n+1}$ diameter of σ.

useful fact: diameter of a simplex = max distance between its vertices.

$\underline{\text{Proof}}$ (of useful fact) $\quad \underline{v}$ any vector and $\sum t_i v_i \in \Delta^n$

$$|v - \sum t_i v_i| = |\sum t_i (v - v_i)| \quad \text{as } \sum t_i = 1$$
$$\leq \sum t_i |v - v_i| \quad \text{triangle inequality}$$
$$\leq \sum t_i \max |v - v_i|$$
$$\leq \max |v - v_i| \quad \text{as } \sum t_i = 1 \quad \square.$$

$\underline{\text{Proof}}$ (of claim): suffices to bound distances between vertices in the barycentric subdivision.

induction on n: base case $n=1$:

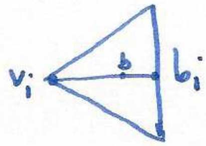
$b = \frac{1}{2}(v_0 + v_1)$ $\quad d(b, v_i) \leq \frac{1}{2} d(v_0, v_1)$ ✓

Let $[w_0, \dots, w_n]$ be a simplex in the barycentric subdivision of $[v_0, \dots, v_n]$

Let w_i, w_j be vertices realizing $\max |w_i - w_j|$

If neither $w_i, w_j = b$, then done by induction, as they live in a proper face of $[v_0, \dots, v_n]$.

wlog $w_j = b$. By convexity may take $w_i = v_i$ for some vertex v_i



Let b_i be the barycenter of the opposite face to v_i, i.e. of $[v_0, \dots, \hat{v}_i, \dots v_n]$

$$b = \sum \frac{1}{n+1} v_i \quad , \quad b_i = \sum_{j \neq i} \frac{1}{n} b_j \quad \text{so} \quad b = \frac{1}{n+1} v_i + \frac{n}{n+1} b_i$$

($\Rightarrow$ b lies in $[v_i, b_i]$). Therefore

$$|v_i - b| \leq \frac{n}{n+1} \underbrace{|v_i - b_i|}_{\text{both points in } \partial[v_0, \dots v_n]}$$

$$\leq \frac{n}{n+1} \operatorname{diam}([v_0, \dots v_n]) \quad \square.$$

note: The diameter of the simplices in the r-th barycentric subdivision of Δ^n is at most $\left(\frac{n}{n+1}\right)^r \operatorname{diam}(\Delta^n) \longrightarrow 0$ as $r \to \infty$.

important: this bound does not depend on the shape of the simplices.

② Barycentric subdivision of linear chains

aim: construct a subdivision map $S : C_n(X) \to C_n(X)$ and show that it is chain homotopic to the identity.

start with linear chains: $\underset{\text{convex}}{Y \subset \mathbb{R}^n}$, consider linear maps $\Delta^n \to Y$

generate a subgroup $\underset{\text{linear chains in } Y}{LC_n(Y)} < \underset{\text{chains in } Y}{C_n(Y)}$

note: $\partial : LC_n(Y) \to LC_{n-1}(Y)$ so $LC_n(Y)$ is a subchain complex of $C_n(Y)$

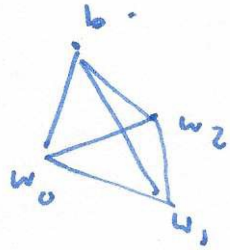
If $\lambda : \Delta^n \to Y$, $\lambda \in LC_n(Y)$, then λ is determined by the images of $[v_0, \dots, v_n]$, call these $w_i = \lambda(v_i)$, so image is $[w_0, \dots, w_n]$

to deal with 0-simplices set $LC_{-1}(Y) = \mathbb{Z}$ generated by $[\phi]$ and $\partial [w_0] = [\phi]$

a point $b \in Y$ determines a <u>cone operator</u>, i.e. a homomorphism

$$b : LC_n(Y) \to LC_{n+1}(Y)$$

$$[w_0, \dots, w_n] \mapsto [b, w_0, \dots, w_n]$$



$$\partial b([w_0, \dots, w_n]) = \sum (-1)^i [b, \dots, \hat{w}_{i-1}, \dots, w_n]$$

$$= [w_0, \dots, w_n] - b\partial [w_0, \dots, w_n]$$

extend linearly over chains:

if $\alpha = \sum n_i \sigma_i$, then $\partial b(\alpha) = \alpha - b\partial\alpha$

i.e. $\quad \partial b(\alpha) + b\partial(\alpha) = \alpha$

i.e. $\quad \partial b + b\partial = \mathbb{1} - 0$

so b is a <u>chain homotopy</u> between $\mathbb{1}$ = identity and 0 map on $LC_n(Y)$

define a subdivision homomorphism $S : LC_n(Y) \to LC_n(Y)$

let $\lambda : \Delta^n \to Y$ be a generator of $LC_n(Y)$

let $b \in \Delta^n$ be the barycenter, $\lambda(b)$ image of b in Y.

<u>define</u>: $\boxed{S(\lambda) = b_\lambda(S\partial\lambda)}$ where $b_\lambda : LC_{n-1}(Y) \to LC_n(Y)$

is the cone operator defined by the image of the barycenter $b(\lambda)$.

<u>note</u>: this is an inductive definition, S on $LC_n(Y)$ defined in terms of S on $LC_{n-1}(Y)$.

<u>base case</u> <u>$n=-1$</u>: $S[\phi] = [\phi]$ so $S = \mathbb{1}$ on $LC_{-1}(Y)$

<u>$n=0$</u>: $S([w_0]) = b_{w_0} S[\phi] = b_{w_0}([\phi]) = [w_0]$

so $S = \mathbb{1}$ on $LC_0(Y)$

<u>$n=1$</u>: $S([w_0,w_1]) = b_\lambda\big(S([w_1]-[w_0])\big) = b_\lambda([w_1]) - b_\lambda[w_0]$.

$= [b,w_1] - [b,w_0]$

and so on....

<u>check</u>: S is a chain map, ie. $\partial S = S\partial$

$LC_n(Y) \xrightarrow{S} LC_n(Y)$

$n=0,1$ $S = \mathbb{1}$ ✓.

general case: $\partial S(\lambda) = \partial\big(b_\lambda(S\partial\lambda)\big)$ (defn of S)

$= (\mathbb{1} - b_\lambda\partial)(S\partial\lambda)$ $(\partial b_\lambda = \mathbb{1} - b\partial_\lambda)$

$= S\partial(\lambda) - b_\lambda(\partial S\partial\lambda)$

$= S\partial(\lambda) - b_\lambda(S\underbrace{\partial\partial}_{=0}\lambda)$ (induction on n!).

$= S\partial(\lambda)$ as required. $\square$.

<u>want</u>: chain homotopy T between S and $\mathbb{1}$

$$\begin{tikzcd}
\cdots \arrow[r] & LC_{k+1}(Y) \arrow[r] \arrow[dl] \arrow[d, "\mathbb{1}"', "S"] & LC_k(Y) \arrow[r] \arrow[dl, "T"'] \arrow[d, "\mathbb{1}"', "S"] & LC_{k-1}(Y) \arrow[r] \arrow[dl, "T"'] \arrow[d, "\mathbb{1}"', "S"] & \cdots \arrow[dl] \\
\cdots \arrow[r] & LC_{k+1}(Y) \arrow[r] & LC_k(Y) \arrow[r] & LC_{k-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$T: LC_n(Y) \to LC_{n+1}(Y)$ $\qquad$ $n=-1$: $T=0$

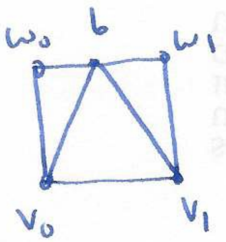
$\qquad\qquad\qquad\qquad\qquad\qquad$ $n \geq 0$: $T\lambda = b_\lambda(\lambda - T\partial\lambda)$

why? subdivide $\Delta^n \times I$ by joining all simplices of $\Delta^n \times \{0\}$ to the corresponding barycenter of $\Delta^n \times \{1\}$, and each face in $T(\partial\Delta^n)$ to corresponding barycenters.

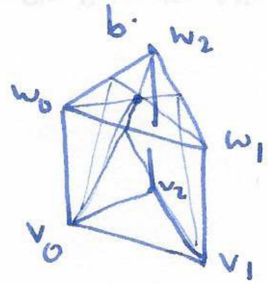
<u>$n=0$</u> :

$T = P$ prism operator

<u>$n=1$</u> :



$n=2$:



<u>check</u>: this is a chain homotopy : $\partial T + T\partial = \mathbb{1} - S$

<u>$n=-1$</u> : $T=0$ $\qquad$ $S = \mathbb{1}$ ✓.

<u>$n \geq 0$</u> : $\partial T\lambda = \partial\left(b_\lambda(\lambda - T\partial\lambda)\right)$ $\quad$ (def^{n} of T)

$= (\mathbb{1} - b_\lambda\partial)(\lambda - T\partial\lambda)$ $\quad$ $(\partial b_\lambda = \mathbb{1} - b_\lambda\partial)$

$= \lambda - T\partial\lambda - b_\lambda\partial(\mathbb{1} - T\partial)\lambda$

$\shortparallel \longleftarrow$ induction on n !

$= \lambda - T\partial\lambda - b_\lambda\partial(S + \partial T)\lambda$

$$= \lambda - T\partial\lambda - b_\lambda \partial S\lambda \cancel{+} - b_\lambda \underbrace{\partial\partial}_{=0} T\lambda$$

$$= \lambda - T\partial\lambda - b_\lambda S\partial\lambda \qquad (\partial S = S\partial)$$

$$= \lambda - T\partial\lambda - S\lambda \qquad (\text{def}^{\underline{n}} \text{ of } S)$$

so we've shown: $\partial T = \mathbb{1} - T\partial - S$

i.e. $\partial T + T\partial = \mathbb{1} - S$. $\square$

③ Barycentric subdivision of general chains. (sketch)

key point: $\Delta^n \xrightarrow{\mathbb{1}} \Delta^n \xrightarrow{\sigma} X$

$\uparrow$ identity is a linear map

aim: $C_n^{LC}(X) \underset{\rho}{\overset{i}{\rightleftarrows}} C_n(X)$

recall for linear chains:

- cone operator $b: [w_0, \dots, w_n] \longrightarrow [b, w_0, w_1, \dots, w_n]$

  $b: LC_n(X) \to LC_{n+1}(X)$ s.t. $\partial b + b\partial = \mathbb{1}$ 

- subdivision operator $S: LC_n(Y) \to LC_n(Y)$

  $S(\lambda) = b_\lambda(S\partial\lambda)$ s.t. $\partial S = S\partial$ 

- subdivided prism operator $T: LC_n(Y) \to LC_{n+1}(Y)$

  $T(\lambda) = b_\lambda(\lambda - T\partial\lambda)$ s.t. $\partial T + T\partial = \mathbb{1} - S$

define $S : C_n(X) \longrightarrow C_n(X)$ i.e. $\Delta \xrightarrow{S} \Delta \xrightarrow{\sigma} X$.

$\sigma \longmapsto \underbrace{\sigma_{\#}}_{\sigma|\text{subsimplex}} \underbrace{S\Delta^n}_{\text{sum of simplices in barycentric subdivision}}$

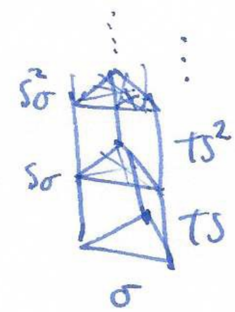
claim: S is a chain map $\partial S = S\partial$ $\square$

define $T : C_n(X) \longrightarrow C_{n+1}(X)$ i.e. $\Delta^n \xrightarrow{T} \Delta^{n+1} \xrightarrow{\sigma} X$.

$\sigma \longmapsto \sigma_{\#} T\Delta^n$

claim: T is a chain homotopy between S and $\mathbb{1}$, i.e. $\partial T + T\partial = S - \mathbb{1}$ $\square$

④ iterated subdivisions



claim $\mathbb{1}$ is chain homotopic to S^m by $D_m = \sum_{i=0}^{m} TS^i$

proof $\partial D_m + D_m \partial = \sum \partial TS^i + TS^i \partial$

$= \sum_i \partial TS^i + T\partial S^i$ (S chain map)

$= \sum_i (\partial T + T\partial) S^i$

$= \sum_i (\mathbb{1} - S) S^i = \mathbb{1} - S + S - S^2 + \cdots - S^m$ $\square$.

for each $\sigma : \Delta^n \to X$ there is an $m(\sigma)$ s.t. $S^{m(\sigma)}\sigma \in C_n^{\mathcal{U}}(X)$, as $\sigma^{-1}(\mathrm{int}(U_i))$ is an open cover of $\Delta^n \subseteq \mathbb{R}^n$, so has a Lebesgue number $\epsilon > 0$. For each σ, let $m(\sigma)$ be the smallest such $m(\sigma)$.

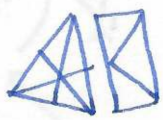
$C_n^{ll}(X) \underset{\rho}{\overset{i}{\rightleftarrows}} C_n(X)$ would like: $\rho(\sigma) = S^{m(\sigma)}\sigma$

doesn't work! (not a chain map).

problem: faces $\sigma_i = \sigma|_{[v_0,\dots,\hat{v}_i,\dots v_n]}$ may have $m(\sigma_i) < m(\sigma)$

(but $m(\sigma_i) \leq m(\sigma)$ for all σ_i)

<u>solution</u>:



<u>define</u>: $\rho(\sigma) = S^{m(\sigma)} - D_{m(\sigma)}\partial\sigma + \sum_{i=0}^{n}(-1)^i D_{m(\sigma_i)}\sigma_i$

[recall: $D_m = \sum_{i=0}^{m} TS^i$ and $\partial D_m + D_m\partial = \mathbb{1} - S^m$]

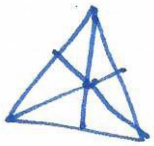
<u>note</u>: $\rho i = \mathbb{1}$ (don't need to subdivide, $\rho(\sigma) = 0$ for all σ).

<u>claim</u>: D is a chain homotopy between $\mathbb{1}$ and $i\rho$

<u>proof</u>:
$$\partial D\sigma + D\partial\sigma = \partial D_{m(\sigma)} + D\sum_{i=0}^{n}(-1)^i\sigma_i$$
$$= \left(\mathbb{1} - S^{m(\sigma)} - D_{m(\sigma)}\partial\right)\sigma + \sum_{i=0}^{n}(-1)^j D_{m(\sigma_i)}\sigma_i$$
$$= \sigma - \underbrace{S^{m(\sigma)}\sigma - D_{m(\sigma)}\partial\sigma + \sum_{i=0}^{n}(-1)^i m(\sigma_i)\sigma_i}_{\rho(\sigma)} \qquad \square.$$

A

① barycentric subdivision of simplices $\Delta^n \subset \mathbb{R}^n$



$$b = \sum_{i=0}^{n} \frac{1}{n+1} v_i \qquad \overset{v_0,\dots,v_n}{[\cancel{w_0,\dots,w_n}]} \rightsquigarrow [b, w_0, \dots w_{n-1}]$$

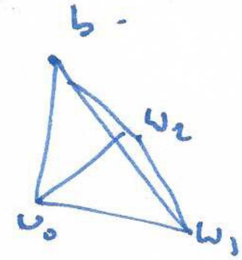
where each $[w_0, \dots w_{n-1}]$ is a face of the barycentric subdivision of a face $[v_0, \dots, \hat{v}_i, \dots v_n]$ of σ.

claim: diam of each subsimplex $\leq \frac{n}{n+1}$ diam original simplex.

note: bound does not depend on the shape of the simplex.

② barycentric subdivision of linear chains

$Y \subset \mathbb{R}^n$ convex $\qquad LC_n(Y) \subset C_n(Y)$ subchain complex



cone operator $b : LC_n(Y) \to LC_{n+1}(Y)$

$$[w_0, \dots, w_n] \mapsto [b, w_0, \dots, w_n]$$

fact: $\partial b + b \partial = 1 - 0$ so b is a chain homotopy between 1 and 0

define $S : LC_n(Y) \to LC_n(Y)$ by

$$S(\lambda) = b_\lambda(S \partial \lambda) \qquad \lambda : \Delta^n \to Y.$$

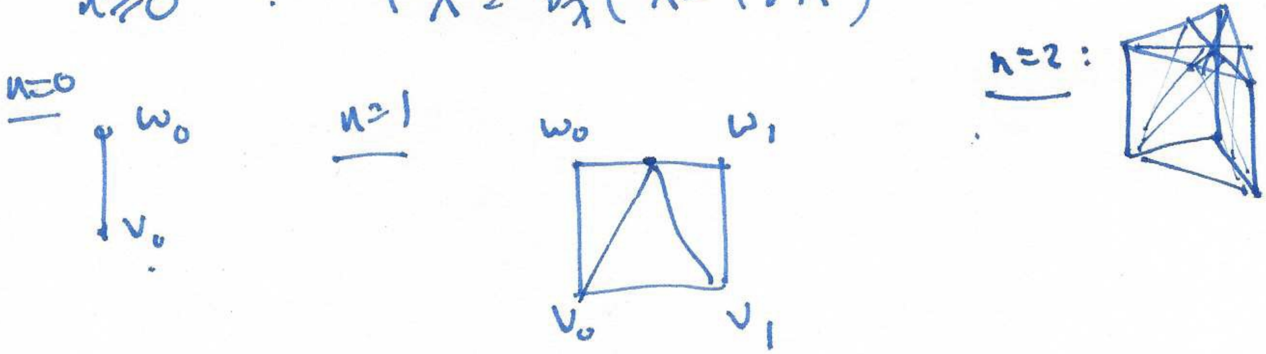
fact: $\partial S = S \partial$ (S is a chain map)

want: chain homotopy T between S and $\mathbb{1}$. (B)

i.e. $\partial T + T\partial = S - \mathbb{1}$

$n = -1, 0$: $T = \sigma P$ prism operator

$n \geq 0$: $T\lambda = b_\lambda(\lambda - T\partial\lambda)$



③ **barycentric subdivision of general chains**

key point : $\Delta^n \xrightarrow{\mathbb{1}} \Delta^n \xrightarrow{\sigma} X$

identity is a linear map

aim : $C_n^{\mathcal{U}}(X) \underset{\rho}{\overset{i}{\rightleftarrows}} C_n(X)$

recall : · cone operator $b : LC_n(Y) \to LC_{n+1}(Y)$ $\quad \partial b + b\partial = \mathbb{1}$

$[w_0, \ldots, w_n] \mapsto [b, w_0, \ldots, w_n]$

· subdivision operator $S : LC_n(Y) \to LC_n(Y)$ $\quad \partial S = S\partial$

$S(\lambda) = b_\lambda(S\partial\lambda)$

· subdivided prism operator $T : LC_n(Y) \to LC_{n+1}(Y)$ $\quad \partial T + T\partial = \mathbb{1} - S$

$T(\lambda) = b_\lambda(\lambda - T\partial\lambda)$

define $S : C_n(X) \to C_n(X)$

$\sigma \longmapsto \underbrace{\sigma_\#}_{\text{restriction to subsimplex}} \underbrace{S\Delta^n}_{\text{sum of simplices in barycentric subdivision}}$

c

<u>claim</u>: S is a chain map $\partial S = S\partial$

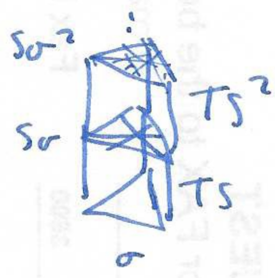
define: $T: C_n(X) \longrightarrow C_{n+1}(X)$ i.e. $\Delta^n \xrightarrow{T} \Delta^{n+1} \xrightarrow{\sigma} X$

$\sigma \longmapsto \sigma_\# T\Delta^n$

<u>claim</u>: T is a chain homotopy between S and $\mathbb{1}$, i.e. $\partial T + T\partial = S - \mathbb{1}$.

④ <u>iterated subdivisions</u>

<u>claim</u> $\mathbb{1}$ is chain homotopic to S^m by $D_m = \sum_{i=0}^{m} TS^i$



i.e. $\partial D_m + D_m \partial = \mathbb{1} - S^m$

for each $\sigma: \Delta^n \to X$ there is an $m(\sigma)$ s.t. $S^{m(\sigma)}\sigma \in C_n^{\mathcal{U}}(X)$

as $\sigma^{-1}(\mathrm{int}(U_i))$ is an open cover of $\Delta^n \subseteq \mathbb{R}^n$, so has a Lebesgue number $\epsilon > 0$.

For each σ, let $m(\sigma)$ be the smallest such σ.

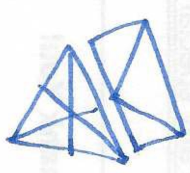
$C_n^{\mathcal{U}}(X) \underset{\rho}{\overset{i}{\rightleftarrows}} C_n(X)$ would like $\rho(\sigma) = S^{m(\sigma)}\sigma$

doesn't work! (not a chain map)

<u>problem</u>: face $\sigma_i = \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$ may have $m(\sigma_i) \lneq m(\sigma)$

(but $m(\sigma_i) \leq m(\sigma)$ for each i)

<u>solution</u>:



<u>define</u>: $\rho(\sigma) = S^{m(\sigma)} - D_{m(\sigma)}\partial\sigma + \sum_{i=0}^{n} (-1)^i D_{m(\sigma_i)}\sigma_i$

<u>claim</u>: $D = D_{m(\sigma)}$ is a chain homotopy between $\mathbb{1}$ and $i\rho$. $\square$.