Example closed non-orientable surface of genus g

cell structure: 1 0-cell

g 1-cells $a_1, \dots, a_g$ gluing map $a_1^2 a_2^2 \cdots a_g^2$

1 2-cell

cellular chain map complex:

$$\mathbb{Z} \to \mathbb{Z}^g \xrightarrow{0} \mathbb{Z}$$

$$1 \mapsto (2,2,\dots,2)$$

$$H_k(M) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z} & k=1 \\ 0 & k=2 \end{cases}$$

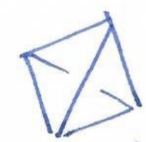
Euler characteristic

recall: surfaces (2-complexes) $\chi(S) = V - E + F$

$V = \#$vertices

$E = \#$edges

$F = \#$faces



$4-6+4=2$

fact: $\chi(S)$ independent of triangulation of the surface.

in general: if X is a CW-complex, then $\chi(X) = \sum_{n=0}^{\infty} (-1)^n c_n$

$c_n = \#$ cells in dimension n.

Thm $\chi(X) = \sum_n (-1)^n \operatorname{rank}(H_n(X))$ (X finite dimensional)

recall: rank $= \#\mathbb{Z}$-summands / dimension of free part / $\dim H_n(X) \otimes \mathbb{Q}$

exercise: $0 \to A \to B \to C \to 0$ exact,

then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

Proof (purely algebraic)

let $0 \to C_k \xrightarrow{d_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$ be a chain complex.

let: cycles: $Z_n = \ker d_n$
boundaries: $B_n = \operatorname{im} d_{n+1}$
homology: $H_n = Z_n / B_n$

then

$$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{d_{n-1}} B_{n-1} \to 0$$

$$0 \to B_n \xrightarrow{i} Z_n \xrightarrow{q} H_n \to 0$$

both short exact

so $\operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}$

$\operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n$

$\Rightarrow \quad \operatorname{rank} C_n = \operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1}$

$\sum (-1)^n \operatorname{rank} C_n = \sum (-1)^n \operatorname{rank} H_n$, as required. $\square$.

Homology of groups

topological space $X \rightsquigarrow H_k(X)$ algebraic invariants of X.

group $G \rightsquigarrow$ topological space $K(G,1) \rightsquigarrow H_k(G) \underset{\text{defn}}{=} H_k(K(G,1))$

$\pi_1(K(G,1)) \cong G$

$\widetilde{X} = \widetilde{K(G,1)}$ contractible

fact X unique up to homotopy equivalence.

invariants of G!

Examples $G = \mathbb{Z} \quad K(G,1) \simeq S^1$

$G = \mathbb{Z}/2\mathbb{Z} \quad K(G,1) \simeq \mathbb{RP}^\infty \leftarrow$ note $H_k \neq 0$ for ∞-many k!

$X =$ fundamental group of M^3 hyperbolic manifold. $\Rightarrow X = K(M, 1)$.