Proof

a) $(X^n, X^{n-1})$ good pair $X^n / X^{n-1} = \bigvee_\alpha S^n_\alpha$, one for each $\alpha$-cell.

b) l.e.s of a pair. $(X^n, X^{n-1})$
c)

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \to H_k(X^n) \to H_k(X^n, X^{n-1}) \to$$

$H_{k+1}(X^n,X^{n-1}) = 0$, ~~$k \geq n$~~ $k \neq n, n-1$; $H_k(X^n,X^{n-1}) = 0$, ~~$k > n$~~ $k \neq n, n-1$

if $k > n$ then $H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \dots \quad H_k(X^0) = 0$

c) if $k < n$ then $H_k(X^n) \cong H_k(X^{n+1}) \cong \dots \cong H_k(X^{n+m})$ for all $m$
$\cong 0$ if $X$ finite dimensional $\square$.

(see Hatcher for infinite dim case).

<u>Cellular chain complex</u>

$$\dots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \dots$$

free abelian. dim = # $n+1$ cells.

defined in terms of degree

$\varphi_\alpha$

$e^2_\alpha$

homology groups of this chain complex are the <u>cellular homology groups</u> $H_n^{CW}(X)$.

$$\begin{tikzcd}
 & & H_n(X^{n-1}) = 0 \arrow[d, "i_n"] & & \\
H_{n+1}^{\dots} \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] \arrow[dr, "d_{n+1}"'] & H_n(X^n) \arrow[r, "i_n"] \arrow[d, "j_n"] & H_n(X^{n+1}) = H_n(X) \arrow[r, "j_n"] & H_n(X^{n+1}, X^n) = 0 \arrow[r] & \dots \\
 & & H_n(X^n, X^{n-1}) \arrow[d, "\partial_n"] \arrow[dr, "d_n"] & & \\
 & H_{n-1}(X^{n-2}) = 0 \arrow[r, "i_{n-1}"'] & H_{n-1}(X^{n-1}) \arrow[r, "j_{n-1}"'] \arrow[d] & H_n(X^{n-1}, X^{n-2}) \arrow[r] & \\
 & & 0 & &
\end{tikzcd}$$

l.e.s pair

l.e.s pair:

exact.

Def$^{\underline{n}}$ $d_{n+1} = j_n \partial_{n+1}$

$d_n = j_{n-1} \partial_n$
etc.

<u>Check</u> $d_n d_{n+1} = 0$: $d_n d_{n+1} = j_{n-1} \partial_n j_n \partial_{n+1}$

Def$^{\underline{n}}$ $\square$.

**Thm** $H_n^{cw}(X) \cong H_n(X)$ if $X$ is a CW-complex.

**Proof** from diagram: $H_n(X) = H_n(X^n) / \text{image}(\partial_{n+1})$.

diagram commutes, $j_n$ injective, so $\text{image}(\partial_{n+1}) \cong \text{image}(j_n \partial_{n+1}) = \text{image}(d_{n+1}')$

also $H_n(X^n) \cong \text{image}(j_n) = \ker(\partial_n)$ $j_{n-1}$ injective

$\Rightarrow = \ker(d_n)$.

therefore $H_n(X) \cong$ ~~ima~~ $\ker(d_n) / \text{image}(d_{n+1})$ $\square$.

**Applications**

1) $H_n(X) = 0$ if $X$ is a CW-complex with no $n$-cells.

2) $X$ CW-complex w/ at most $k$ $n$-cells, then $H_n(X)$ has at most $k$ generators.

3) if $X$ has no cells in adjacent dimensions, then $H_n(X)$ is free abelian, with basis the cells.

**Examples** $\mathbb{C}P^n$ (one cell in each even dim)

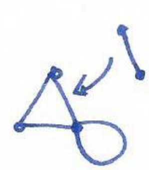
$S^n$

$S^n \times S^n$

**Computing the boundary map**

$n=1$ $d_1 : H_1(X', X^0) \longrightarrow H_0(X^0)$ same as simplicial boundary map

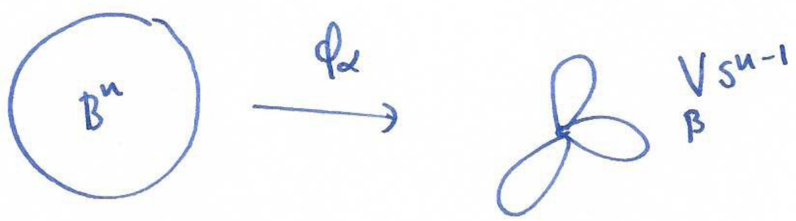
$H_1^\Delta(X) \longrightarrow H_0^\Delta(X)$

$X^{(0)}$  $X^{(1)}$ 

special case: if $X$ connected with 1 vertex then $d_1 = 0$

$\underline{n \geq 2}$ : $d_n : (X^n, X^{n-1}) \longrightarrow H_n(X^{n-1}, X^{n-2})$

$n$-cell $e^n_\alpha$




$\partial B^n = S^{n-1}$

$$\varphi_\alpha : \underset{S^{n-1}}{\overset{}{\partial B^n}} \longrightarrow \bigvee_\beta S^{n-1} \xrightarrow{p_\beta} S^{n-1}_\beta$$

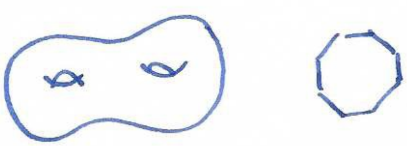
$$(X^{n-1}, X^{n-2}) \qquad (X^{n-1}, X^{n-1} \setminus S^{n-1}_\beta)$$

<u>cellular boundary formula</u>

$$d_n(e_\alpha) = \sum_\beta d_{\alpha\beta}\, e_\beta^{n-1}$$

where $d_{\alpha\beta}$ is the degree of the map $S^{n-1} \xrightarrow[\varphi_\alpha]{} X^{n-1} \xrightarrow[q]{} S_\beta^{n-1}$
given by composition of the gluing map
with the quotient map collapsing $X^{n-1} \setminus e_\beta^{n-1}$ to a point

<u>Example</u> $M_g$ closed orientable surface of genus $g$



1 0-cell
$2g$ 1-cells $a_1, b_1, a_2, b_2, \dots, a_g, b_g$
1 2-cells

gluing map : $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$

chain complex: $\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow[0]{d_1} \mathbb{Z}$

each $a_i$, $b_i$ appears exactly twice with opposite signs.
$\Rightarrow d_2 = 0$.

so $H_k(M_g) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{2g} & k=1 \\ \mathbb{Z} & k=2 \end{cases}$