<u>How to compute degree</u>: pick generic $x \in S^n$ and count $f^{-1}(x)$ with signs:

$\deg(f) = 1 - 1 + 1 = 1$

non-generic points:

Let $f : S^n \to S^n$ s.t. for some $y \in S^n$ $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_k$. Let $U_1, \dots, U_k$ be disjoint open neighbourhoods of $x_1, \dots, x_k$ s.t. $f(U_i) \subset V$, some open neighbourhood of y, s.t. $f(U_i \setminus x_i) \subset V \setminus y$ for each i. Write $f_*|_{x_i}$ for the local degree map $H_n(U_i, U_i \setminus x_i) \to H_n(V, V \setminus y)$. (if f is a local homeomorphism, then $f_* = \pm 1$)

<u>Prop$^{\underline{n}}$</u> $\deg(f) = \sum_{i=1}^{k} \deg(f|_{x_i})$

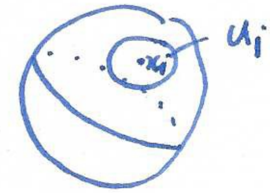
$$\begin{tikzcd}
 & \overset{\mathbb{Z}}{H_n(U_i, U_i \setminus x_i)} \arrow[dl, "\cong"', "\text{excision}"] \arrow[d, "k_i"', "i_*"] \arrow[r, "(f_*|_{x_i})_*"] & \overset{\mathbb{Z}}{H_n(V, V \setminus y)} \arrow[d, "i_*"', "\cong \text{ (excision)}"] \\
\underset{\mathbb{Z}}{H_n(S^n, S^n \setminus x_i)} & \underset{\mathbb{Z}^{|f^{-1}(y)|}}{H_n(S^n, S^n \setminus f^{-1}(y))} \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y)\ \mathbb{Z}. \\
 & \underset{\mathbb{Z}}{H_n(S^n)} \arrow[ul, "\cong", "\text{l.e.s of a pair}"'] \arrow[u, "j", "\text{l.e.s. of a pair}"'] \arrow[r, "f_*"', "\text{degree map.}"'] & \underset{\mathbb{Z}.}{H_n(S^n)} \arrow[u, "\cong \text{ l.e.s. of a pair}"']
\end{tikzcd}$$

claim: diagram commutes.

<u>note</u> $H_n(S^n, S^n \setminus f^{-1}(y)) \underset{\substack{\uparrow \\ \text{excision}}}{\cong} H_n(\amalg U_i, x_i) \cong \bigoplus \underbrace{H_n(U_i, x_i)}_{\cong\, \mathbb{Z}}$.

diagram commutes:

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, "\text{excision}"', "\cong"] \arrow[d, "k_i\ \text{inclusion}"] \\ \mathbb{Z} & \mathbb{Z}^k \arrow[l, "p_i"', "\text{projection}"] \end{tikzcd}$$



$H_n(S^n \setminus U_i^c, S^n \setminus x_i \setminus U_i^c) = H_n(U_i, U_i \setminus x_i)$

$$\begin{tikzcd} \mathbb{Z} & \mathbb{Z}^k \arrow[l, "p_i"'] \\ & \mathbb{Z} \arrow[u, "j"] \arrow[ul, "\cong"] \end{tikzcd}$$

l.e.s pair

long exact sequence of a pair $(S^n, S^n \setminus f^{-1}(y))$

$$H_n(S^n) \xrightarrow{j} H_n(S^n, S^n \setminus f^{-1}(y)) \cong \bigoplus H_n(U_i, U_i \setminus x_i).$$

$$\mathbb{Z} \ni \longrightarrow \mathbb{Z}^k$$

$$1 \longmapsto (1, 1, \ldots, 1)$$

$$\begin{array}{ccccccc} H_n(S^n \setminus x_i) & \to & H_n(S^n) & \to & H_n(S^n, S^n \setminus x_i) & \to & H_{n-1}(S^n \setminus x_i) \\ 0 & & \mathbb{Z} & \cong & \mathbb{Z} & & 0 \end{array}$$

$$\begin{array}{ccc} 1 & \longmapsto & \deg(f|x_i) \\ \mathbb{Z} & \xrightarrow{(f_*|x_i)_*} & \mathbb{Z} \\ \downarrow i_{k*} & & \downarrow i_* \\ \mathbb{Z}^k & \xrightarrow{f_*} & \mathbb{Z} \\ (0, \ldots, 1, 0 \ldots) & \longmapsto & f_*(i_{k*}(1)) \end{array} \qquad \deg(f|x_i) = f_*(i_{k*}(1)).$$

$\leftarrow$ commutes because

$$\begin{tikzcd} X \arrow[r, "f|"] \arrow[d, hook, "i"] & Y \arrow[d, hook, "i"] \\ X^+ \arrow[r, "f"] & Y^+ \end{tikzcd}$$

$\sigma \mapsto f(\sigma)$, $i(\sigma)$, $if(\sigma)$, $f(i(\sigma))$.

this commutes. (at the chain level).

$$\begin{array}{ccc} (1, \ldots, 1) & \longmapsto & \sum \deg(f|x_i) \\ \mathbb{Z}^k & \xrightarrow{f_*} & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \xrightarrow{f_*} & \mathbb{Z} \\ 1 & \longmapsto & \deg(f) \end{array} \qquad \sum \deg(f|x_i) = \deg(f)$$

$\checkmark$ $\square$.

$\leftarrow$ commutes because l.e.s of a pair is <u>natural</u>.

$(A, B) \xrightarrow{f} (C, D)$

then

$$\begin{array}{ccccccc} \cdots \to & H_n(B) & \to & H_n(A) & \to & H_n(A, B) & \to \cdots \\ & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & \\ \cdots \to & H_n(D) & \to & H_n(C) & \to & H_n(C, D) & \to \cdots \end{array}$$

commutes. (at chain level).

Example $z \mapsto z^d$ gives a map of degree d on S^1 Prop $\stackrel{n}{=}$.

suspend this to get maps of degree d on S^n. (i.e. $\deg(Sf) = \deg(f)$).

check this: (long exact sequence of a pair and naturality)

pair: $(S^n \times I, S^n \times \partial I)$ $\qquad \partial I = \{0,1\}$. $\qquad H_k(S^n \times I, S^n \times \partial I) \cong H_k(S^{n+1})$.

$$
\begin{array}{ccccccccc}
H_{n+1}(S^n \times I) & \xrightarrow{j} & H_{n+1}(S^n \times I, S^n \times \partial I) & \xrightarrow{\partial} & H_n(S^n \times \partial I) & \xrightarrow{i} & H_n(S^n \times I) & \longrightarrow & H_n(S^n \times I, S^n \times \partial I) \\
0 & & \mathbb{Z} & 1 \mapsto (1,-1) & \mathbb{Z} \oplus \mathbb{Z} & (a,b) \mapsto a+b & \mathbb{Z} & & 0 \\
f_* \downarrow & & \downarrow \deg(Sf) & & \downarrow (\deg(f), \deg(f)) & & \downarrow \deg(f) & & \downarrow f_* \\
0 & & \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} & & 0
\end{array}
$$

$\deg(Sf) \longmapsto (\deg(Sf), -\deg(Sf))$

$f : S^n \to S^n$ $\qquad f : (S^n \times I, S^n \times \partial I) \longrightarrow (S^n \times I, S^n \times \partial I)$.

$$S^n \times I \xrightarrow[f \times id_I]{} S^n \times I$$

$$S^n \times \partial I \xrightarrow[f \amalg f]{} S^n \times \partial I.$$

$\square$.

Cellular homology

X CW-complex. $\qquad$ **example** $S^n =$  1 n-cell $\cup$ 1 0-cell.

useful facts.

a) $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$

$k = n$ free abelian with basis n-cells of X.

b) $H_k(X^n) = 0$ for $k > n$

c) $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \longrightarrow H_k(X)$ for all $k < n$.