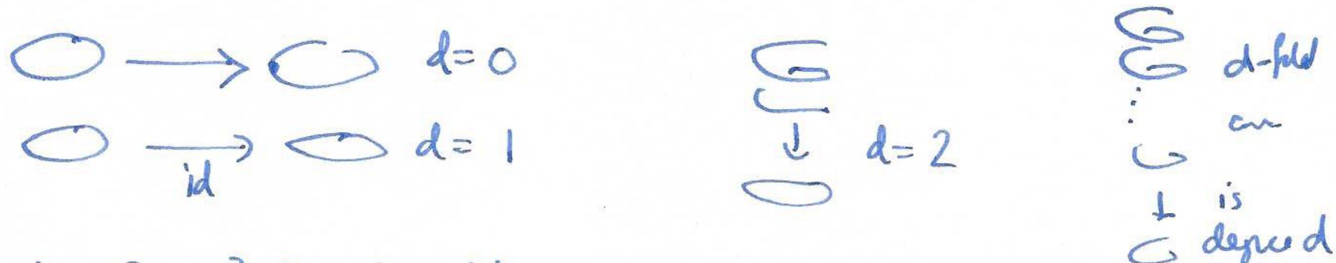
§2.2 Applications

Defn A map $f: S^n \to S^n$ induces $f_*: H_n(S^n) \to H_n(S^n)$

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \cong & & \cong \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

so $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$. This integer is called the degree.

Examples



Q: what about $S^2 \to S^2$? A: take suspensions...

useful properties

a) $\deg(\underset{\mathbb{1}}{id}) = 1$ as $\mathbb{1}_* = \mathbb{1}$.

b) $\deg(f) = 0$ if f not surjective, can factor $S^n \to \underbrace{S^n \setminus \{x\}}_{\text{contractible}} \to S^n$

so $H_n(S^n \setminus \{x\}) = 0$.

c) $f \simeq g \Rightarrow \deg(f) = \deg(g)$. $(f_* = g_*)$

d) $\deg(f \circ g) = \deg(f)\deg(g)$ $\quad ((fg)_* = f_* g_*)$

in particular, if f is a homeomorphism or homotopy equivalence, then $\deg(f) = \pm 1$.

e) $f: S^n \to S^n$ reflection in S^{n-1}, then $\deg(f) = -1$

Proof: $S^n = \Delta_1^n \cup \Delta_2^n$ $\quad [v_0, \dots, v_n] \leftrightarrow [w_0, \dots, w_n]$

$H_n(S^n)$ generated by $\Delta_1 - \Delta_2$. Reflection: $\Delta_1 - \Delta_2 \mapsto \Delta_2 - \Delta_1$.

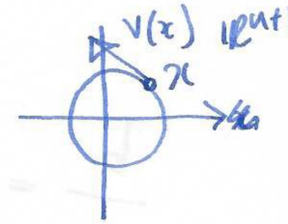
f) the antipodal map $-\mathbb{1}: S^n \to S^n$ has degree $(-1)^{n+1}$, as it is a composition of $(n+1)$ reflections.

g) if $f: S^n \to S^n$ has no fixed points, then f is homotopic to $-\mathbb{1}$

by $f_t(x) = \dfrac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$ so $\deg(f) = (-1)^{n+1}$

<u>Thm</u> S^n has a continuous field of non-zero tangent vectors iff n is odd.

<u>Proof</u> suppose $x \mapsto v(x)$ is a tangent vector field on S^n.



(i.e. x and $v(x)$ are orthogonal)

if $v(x) \neq 0$ then we can normalize $\dfrac{v(x)}{\|v(x)\|}$

consider: $v_t(x) = (\cos t)x + (\sin t)v(x)$ is a homotopy from the identity map $\mathbb{1}$ to the antipodal map $-\mathbb{1}$.

so $\underset{\substack{\| \\ 1}}{\deg(\mathbb{1})} = \underset{\substack{\| \\ (-1)^{n+1}}}{\deg(-\mathbb{1})} \Rightarrow n$ odd $\square$.

<u>Exercise</u> construct $v(x)$ on S^1, S^3.

<u>Applications</u>

<u>Propn</u> $\mathbb{Z}/2\mathbb{Z}$ is the only group which can act freely on S^n if n is even.

(G acts on $S^n \Leftrightarrow$ there is an (injective) homomorphism $G \to \mathrm{Homeo}(S^n)$).

<u>Proof</u> $\deg(f) = \pm 1$ if $f \in \mathrm{Homeo}(S^n)$, so we get $G \to \mathrm{Homeo}(S^n) \xrightarrow{d} \mathbb{Z}/2\mathbb{Z}$ homomorphisms. If G acts <u>freely</u>, i.e. no fixed points, then $d(f) = (-1)^{n+1} = -1$ ~~for all~~ (n even) for all $f \neq \mathbb{1}_G$, so $\ker(d) = \{1\}$

$\Rightarrow G \cong \mathbb{Z}/2\mathbb{Z}$ $\square$.