Topology I Math 70800 (second semester)

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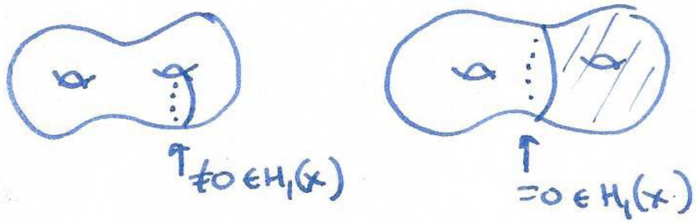
Text: Algebraic Topology, Allen Hatcher.

§2 Homology

<u>intuition</u>: $H_n(X) =$ "n-dim objects w/ no boundary" / "n-dim objects which bound (n+1)-dim objects".

<u>example</u>:



<u>simplices</u> $[v_0]$ • $[v_0, v_1]$

$[v_0, v_1, v_2]$

<u>boundary map</u>

$\partial : [v_0] \longmapsto \phi$.

$\partial : [v_0, v_1] \longmapsto [v_1] - [v_0]$

$\partial : [v_0, v_1, v_2] \longmapsto [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$.

$\partial : [v_0, \dots, v_n] \longmapsto \sum_{i=0}^{n} (-1)^i [v_0, \dots \hat{v}_i \dots v_n]$.

<u>Example</u> simplicial homology: $X =$ simplicial complex Δ-complex $\sigma_\alpha : \Delta_\alpha \to X$.

chain groups $\Delta_n(X) =$ formal sums of n-simplices in X

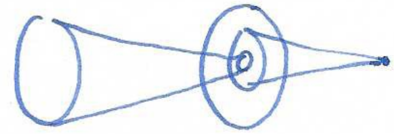
$= \sum_\alpha n_\alpha \sigma_\alpha$

<u>Def$^{\underline{n}}$</u> A <u>chain complex</u> is a sequence of abelian groups $\cdots \Delta_{n+1}(X) \xrightarrow{\partial} \Delta_n(X) \xrightarrow{\partial} \Delta_{n-1}(X) \to \cdots$ with homomorphisms $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ s.t. $\partial^2 = 0$.

<u>key fact</u> $\partial^2 = 0$.

Def^n The <u>n-th homology group</u> of a chain complex is

$$H_n = \ker(\partial_n) / \mathrm{im}(\partial_{n+1}). \qquad C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$



where
$C_n = \Delta_n(X)$
Then $H_n^\Delta(X)$ = n-th simplicial homology group of X.

Example singular homology: X = topological space

$C_n(X)$ = formal sums of maps $\sigma_\alpha : \Delta^n \to X$.

$= \sum n_\alpha \sigma_\alpha$

boundary map: $\partial_n : C_n(X) \to C_{n-1}(X)$

$$\sigma \longmapsto \partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}.$$

note $\partial_n \partial_{n+1} = 0$

$H_n(X)$ is the n-th singular homology group of X.

Thm If X is a simplicial complex, then $H_n^\Delta(X) \cong H_n(X)$ for all n.

Induced maps $f : X \to Y$. $\quad \sigma : \Delta^n \to X \longmapsto f_\# \sigma : \Delta^n \to Y$
$f \circ \sigma$

Def^n A <u>chain map</u> between two chain complexes is:

$$\begin{array}{ccccccc}
\cdots \to C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \to \cdots \\
\downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & \\
C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \to \cdots
\end{array}$$

s.t. <u>commute</u> i.e. $\partial f_\# = f_\# \partial$.

Prop^n A chain map induces homomorphisms $f_* : H_n(X) \to H_n(Y)$ for all n.

Thm if $f \simeq g$ then $f_* = g_*$.

Corollary if $X \underset{h.e.}{\simeq} Y$ then $H_*(X) = H_*(Y)$.

Exact sequences and excision

a sequence of abelian groups and homomorphisms is <u>exact</u> if

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots \quad \text{if} \quad \mathrm{im}(d_{n+1}) = \ker(d_n) \text{ for all } n.$$

(i.e. chain complex has trivial homology).

<u>observations</u>:

$0 \longrightarrow A \xrightarrow{\alpha} B$ exact $\Rightarrow$ α injective

$A \xrightarrow{\alpha} B \longrightarrow 0$ exact $\Rightarrow$ α surjective

$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ exact $\Rightarrow$ α isomorphism

$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ exact iff α injective, β surjective, $\ker \beta = \mathrm{im}\, \alpha$

short exact sequence

$C \cong B/\mathrm{im}(\alpha) \cong B/A$. ← <u>warning</u> don't ever write this!

Q: what is $\mathbb{Z}/\mathbb{Z}$? $\mathbb{Z}/n\mathbb{Z}$?

<u>excision</u> $A \subset X$ can construct quotient space X/A "squash A to a point"

$X/A = X \setminus A \amalg \{pt\}$ ← open sets: open sets in $X \setminus A$, open sets <u>containing</u> $A \cup \{pt\}$.

observation: if A subcomplex of X then can just squash A to single 0-cell.

<u>Defn</u> (X, A) is a <u>good pair</u>.

<u>example</u> $\partial D^2 \subset D^2$  $D^2/\partial D^2 = S^2$

<u>Thm</u> $A \subset X$ non-empty, closed, deformation retract of an open set.

Then there is an exact sequence:

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(X/A) \to \cdots$$

where i_* and j_* are induced by the maps $A \xhookrightarrow{i} X \xrightarrow{j} X/A$ and ∂ is the boundary map.

boundary map:

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_n(A) & \xrightarrow{i_\#} & C_n(X) & \xrightarrow{j_\#} & C_n(X,A) & \to & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \to & C_{n-1}(A) & \xrightarrow{i_\#} & C_{n-1}(X) & \xrightarrow{j_\#} & C_{n-1}(X,A) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

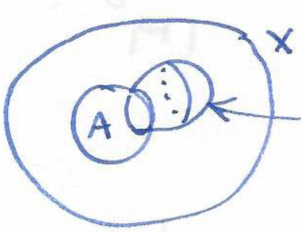
$\leftarrow$ short exact

$b \mapsto c$, $a \mapsto \partial b$

define: $\partial : H_n(X,A) \to H_{n-1}(A)$ check: well defined/homomorphism. (diagram chase)

$[c] \longmapsto [a]$

intuition:



$\partial c = 0$ in $C_n(X,A)$

$\Rightarrow \partial c \in C_{n-1}(A)$. i.e. $\partial[\alpha] = [\partial\alpha]$.

long exact sequence of a pair: (X,A)

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots \quad \text{exact}$$

long exact sequence of a triple: (X,A,B)

$$\cdots \to H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots \quad \text{exact.}$$

Excision $Z \subset A \subset X$ (closed) $\bar{Z} \subset \text{int}(A)$ then $H_n(X \setminus Z, A \setminus Z) \xrightarrow[\cong]{i_*} H_n(X,A)$.

equivalently, if $A \cup B = X$ (open cover) then $H_n(X,A) \xleftarrow[\cong]{i_*} H_n(B, A \cap B)$.

Corollary wedge sum $\bigvee_\alpha X_\alpha$ where each pair (X_α, x_α) is good,

then $\bigoplus_\alpha i_\alpha : \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\bigvee_\alpha X_\alpha)$ is an isomorphism.

Corollary (of $H_*(X) \cong H_*^\Delta(X)$) if X is cw-complex with finitely many cells in each dimension, then $H_*(X)$ is finitely generated.