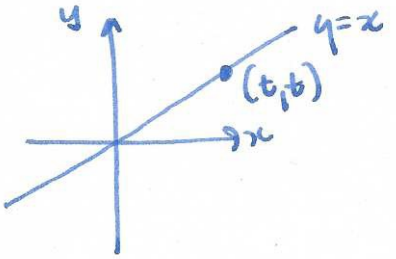
$$\int_0^\infty 2\pi e^{-x}\sqrt{1+(-e^{-x})^2}\,dx \qquad \text{sub} \quad u = e^{-x} \quad \frac{du}{dx} = -e^{-x}$$

$$\int_1^0 2\pi e^{-x}\sqrt{1+u^2}\,\frac{1}{-e^{-x}}\,du = \int_0^1 2\pi\sqrt{1+u^2}\,du \quad (\text{trig sub. integral})$$

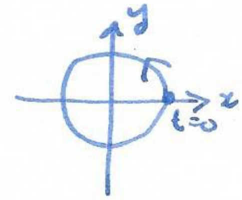
§11.1 Parametric equations



$x(t) = t$ $\qquad c(t) = (x(t), y(t))$

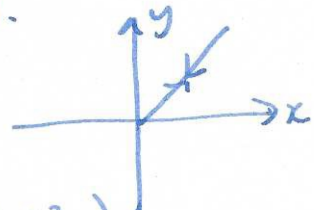
$y(t) = t$ $\qquad \uparrow$ much more flexible,

eg. $c(t) = (\cos t, \sin t)$



problem many different parameterizations give the same curve.

eg $\quad c(t) = (\cancel{2x(t)}\ 2t, \cancel{2y(t)}\ 2t) \qquad c(t) = (t^3, t^3).$



warning : $c(t) = (t^2, t^2)$ not the same

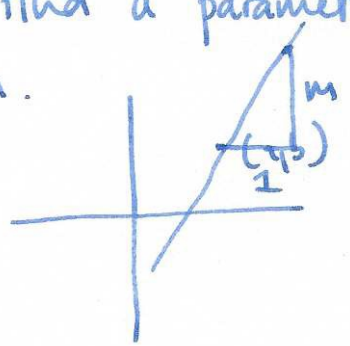
Example describe the parametric curve $c(t) = (t+1, t^2+1)$ in terms of $y = f(x)$.

$x = t+1 \Rightarrow t = x-1$

$y = t^2+1 \qquad y = (x-1)^2+1$

$\qquad\qquad = x^2-2x+2$

Example ① find a parametric equation for the line through (a,b) with slope m.

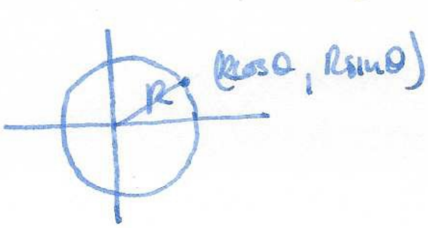


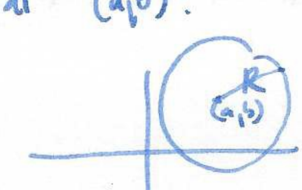
$t=0: (a,b)$

$t=1: (a+1, b+m)$

$c(t) = (a+t, b+mt)$

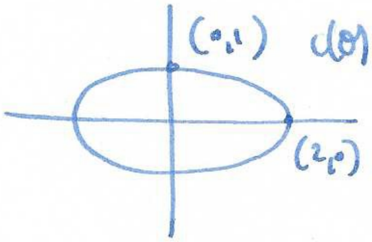
② circle of radius R, centered at (a,b).





$c(\theta) = (a + R\cos\theta, b + R\sin\theta)$

③ ellipse



$c(\theta) = (2\cos\theta, \sin\theta)$

<u>Q</u>: how do we find the slope of the tangent line?

<u>A</u>: $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}}$

<u>why</u>: $c(t) = (x(t), y(t))$

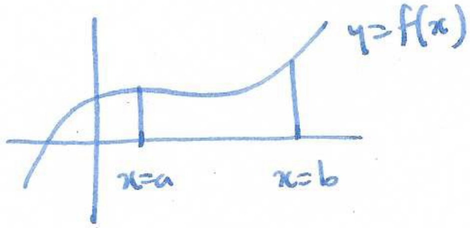
chain rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} \Rightarrow \dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$.

<u>Example</u> $c(t) = (\cos t, \sin 2t)$

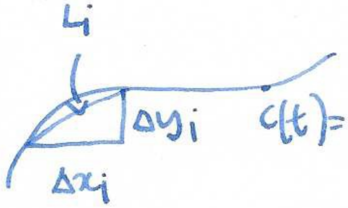
$$\left.\begin{array}{ll} x(t) = \cos t & y(t) = \sin 2t \\ \dfrac{dx}{dt} = -\sin t & \dfrac{dy}{dt} = 2\cos 2t \end{array}\right\} \dfrac{dy}{dx} = \dfrac{2\cos 2t}{-\sin t}$$

<u>§11.2 Arc length and speed</u>

<u>recall</u>



$$\text{arc length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$c(t) = (x(t), y(t))$

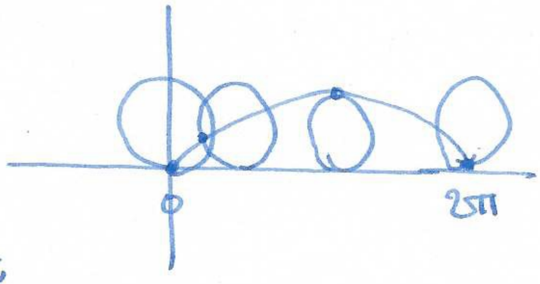
$$\text{arc length} = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

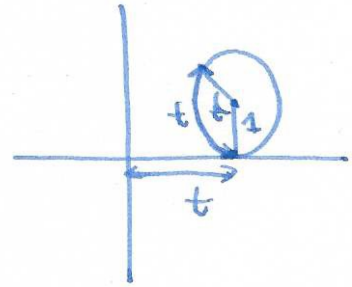
$$= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\, \Delta t_i$$

take limit as $|\Delta t_i| \to 0$

$$\text{arc length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example cycloid





$$x(t) = t - \sin t$$
$$y(t) = 1 - \cos t$$

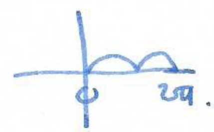
$$\frac{dx}{dt} = 1 - \cos t \qquad \frac{dy}{dt} = \sin t$$

arc length $= \displaystyle\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\; dt$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\; dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\; dt$$

recall: $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\qquad = 1 - 2\sin^2\theta$
$\cos t = 1 - 2\sin^2 \frac{t}{2}$
$2\sin^2\frac{t}{2} = 1 - \cos t$

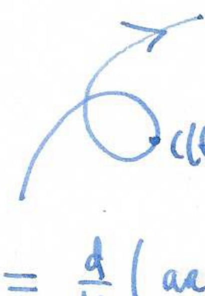
$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{t}{2}}\; dt.$$



$$= \int_0^{2\pi} 2\left|\sin\frac{t}{2}\right| dt = 4\int_0^{\pi} \sin\frac{t}{2}\, dt$$

$$= 4\left[-\frac{\cancel{4}\,2}{\cancel{8}}\cos\frac{t}{2}\right]_0^{\pi} = 8(0+1) = 8.$$

speed



$c(t) = (x(t), y(t))$

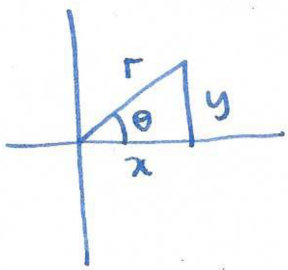
$$\text{speed} = \frac{d}{dt}(\text{arc length}) = \frac{d}{dt}\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

surface area of rotated parameterized curve

surface area $= 2\pi \displaystyle\int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$

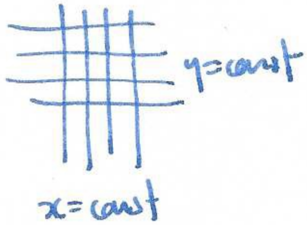
example: $c(t) = (t - \tanh t, \operatorname{sech} t)$

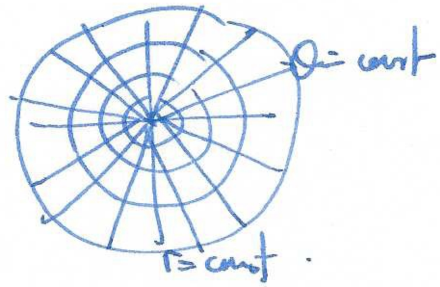
§11.3 Polar coordinates



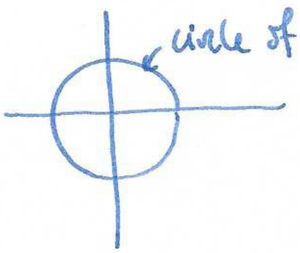
$x = r\cos\theta \qquad \tan\theta = y/x$

$y = r\sin\theta \qquad r^2 = x^2 + y^2$



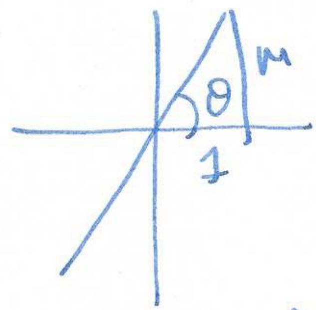


Equations in polar coordinates



circle of radius R: cartesian $x^2+y^2=R^2$

polar $r=R$



straight line through origin

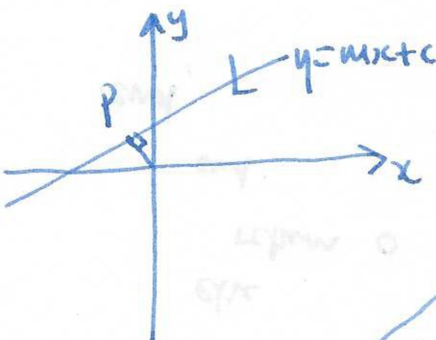
$y = mx$

$\theta = \tan^{-1}(m)$

$\theta = \tan^{-1}(m) + \pi$

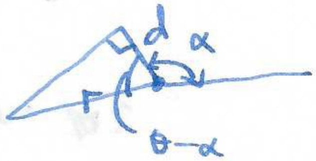
convention: $(-r,\theta) = (r, \theta+\pi)$

so just need $\theta = \tan^{-1}(m)$.



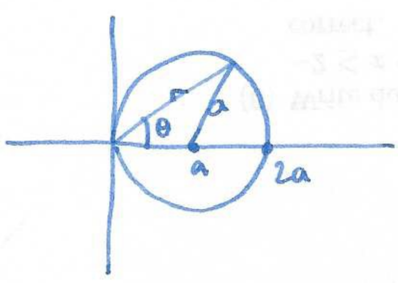
in polars: let P be closest point on L to origin.

let $d(P, \underline{0}) = d$



$$\frac{d}{r} = \cos(\theta-\alpha)$$

$$r = \frac{d}{\cos(\theta-\alpha)} = d\sec(\theta-\alpha).$$



cartesian: $(x-a)^2 + y^2 = a^2$

polar: $r = 2a\cos\theta$

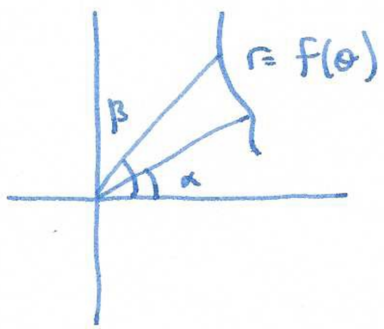
cosine rule:

$a^2 = r^2 + a^2 - 2ar\cos\theta$

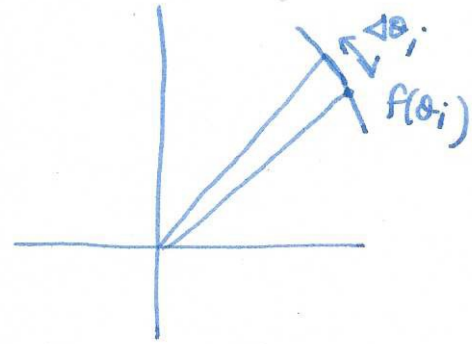
$r^2 = 2ar\cos\theta$

$r = 2a\cos\theta$.

§11.4 Area and arc length in polars.

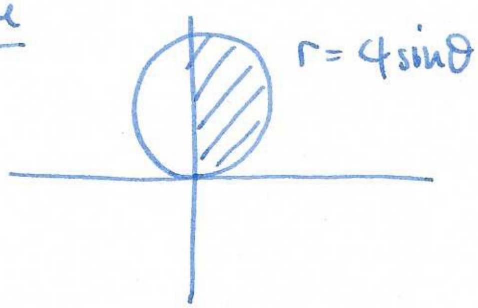


$$\boxed{\text{area} = \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta}$$



area $\Delta A \approx \dfrac{\pi r^2}{2\pi/\Delta\theta_i} = \frac{1}{2} r^2 \Delta\theta_i$

Example

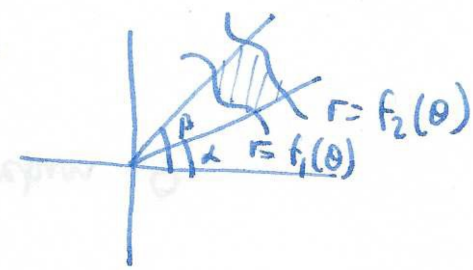


$$\text{area} = \int_0^{\frac{\pi}{2}} \frac{1}{2}(4\sin\theta)^2 d\theta$$

$$= \int_0^{\pi/2} 8\sin^2\theta \, d\theta$$

$$= 8\int_0^{\pi/2} \frac{1}{2}(1-\cos 2\theta)\, d\theta = \left[4\theta - 2\sin 2\theta\right]_0^{\pi/2} = 4 \cdot \frac{\pi}{2} - 0 = 2\pi.$$

area between two curves:



$$\frac{1}{2}\int_{\alpha}^{\beta} (f_2(\theta))^2 - (f_1\theta)^2 d\theta.$$