<u>Example</u> show $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges.

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}} \right|$

$= \lim_{n\to\infty} \left| \left(1+\frac{1}{n}\right)\frac{1}{3} \right| = \frac{1}{3} < 1.$

<u>Bad example</u> $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$

<u>Thm</u> (Root test) $\{a_n\}$ sequence, suppose that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists

① if $L < 1$ then $\sum a_n$ converges absolutely

② if $L > 1$ then $\sum a_n$ diverges

③ if $L = 1$ no information.

<u>Example</u> $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

<u>§10.6 Power series</u>

<u>Defn</u> A power series centered at $x=a$ is an infinite sum of the form $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$

<u>note</u>: this gives a function of x, <u>if the series converges</u>.

the series always converges for $x=a$! $F(a) = a_0$.

<u>Thm</u> <u>Radius of convergence</u> Let $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ then

① $F(x)$ converges <u>only</u> for $x=a$ $(R=0)$

or ② $F(x)$ converges for all x $(R=\infty)$

③ there is a number $R>0$ s.t. $F(x)$ converges absolutely for $|x-a|<R$ and diverges for $|x-a|>R$. It may or may not converge for $x = a \pm R$.

we call R the <u>radius of convergence</u>.

<u>Example</u> Q: for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{2}\right|$

so converges for $\left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$ (so $R = 2$)

<u>Example</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-5)^n$ ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{n+1}\frac{n}{(x-5)^n}\right|$

$= \lim_{n\to\infty}\left|\frac{n}{n+1}(x-5)\right| = |x-5|$ so converges for $|x-5| < 1$

($R=1$).

<u>Examples</u> $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} n!\, x^n$.

<u>Thm</u> Suppose $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of convergence $R > 0$

then $F(x)$ is differentiable on $(-R-a, a+R)$ and furthermore:

$$\left.\begin{aligned} F'(x) &= \sum_{n=1}^{\infty} a_n\, n\,(x-a)^{n-1} \\ \int F(x)\,dx &= \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n} + C \end{aligned}\right\} \text{same radius of convergence } R$$

<u>Example</u> ① $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ radius of convergence $R=1$.

$$\left.\begin{aligned} &\text{so } \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &\text{and } -\ln|1-x| = c + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \end{aligned}\right\} \text{radius of convergence } R=1$$

② $1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2}$

so $\tan^{-1}(x) = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

③ Fact: $y = e^x$ is the (unique) solution to $\frac{dy}{dx} = y$

$f'(x) = f(x)$.

$$\frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

so $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ radius of convergence $R = \infty$.

How to find a power series solution to a differential equation:

try $y = a_0 + a_1 x + a_2 x^2 + \cdots$

__Example__: $y' = y$: $a_1 + 2a_2 x + 3a_3 x^2 + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots$

$a_1 = a_0$

$2a_2 = a_1 = a_0$ $\quad a_2 = \frac{a_0}{2}$

$3a_3 = a_2$ $\quad a_3 = \frac{a_0}{2.3} = \frac{a_0}{3!}$

etc.

__Example__ $y'' + y' + y = 0$ with $y(0) = 1$, $y'(0) = 1$.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$
$$+ 2a_2 + 6a_3 x + \cdots$$

$a_0 = 1$

$a_1 = 1$

$a_0 + a_1 + 2a_2 = 0 \Rightarrow a_2 = -1$

$a_1 + 2a_2 + 6a_3 = 0 \Rightarrow a_3 = \frac{-1+2}{6} = \frac{1}{6}$

etc.

__§10.7 Taylor series__

suppose $f(x)$ has a power series expansion at $x = c$, i.e.

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

__Q__: how do we find the a_i?

__A__: $f(c) = a_0$

$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$ $\quad f'(c) = a_1$

$f''(x) = 2a_2 + 6a_3(x-c) + \cdots$ $\quad f''(c) = 2a_2$