similarly $\frac{b_n}{a_n} \to \frac{1}{L}$ so $0 < \frac{b_n}{a_n} < R'$ $\quad \frac{1}{L} < R'$

$$0 < b_n < R' a_n$$

so comparison test $\Rightarrow$ $\sum a_n$ converges $\Rightarrow$ $\sum b_n$ converges

(if $L=0$ only get one direction). $\square$

Example show $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ converges. (for large n, $a_n \sim \frac{1}{n^2}$).

compare with $b_n = \frac{1}{n^2}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n^2}{n^4-n-1}}{\frac{1}{n^2}} = \frac{n^4}{n^4-n-1} = 1$

so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{n^2}{n^4-n-1}$ converges.

Example does $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2-9}}$ converge? compare with $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2-9}}}{1/n} = \frac{n}{\sqrt{n^2-9}} = \frac{1}{\sqrt{1-9/n^2}} = 1$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

§10.4 Absolute and conditional convergence

Q: what about $\frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ (*)

Def$^{\underline{n}}$ A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

so (*) is absolutely convergent.

Example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots$ not absolutely convergent.

Th^m Absolute convergence $\Rightarrow$ convergence

Proof $0 \leq a_n + |a_n| \leq 2|a_n|$

$$\underbrace{\sum_{n=1}^{\infty} 2|a_n| = 2\sum_{n=1}^{\infty} |a_n|}_{\text{converges}} \Rightarrow \underbrace{\sum_{n=1}^{\infty} a_n + |a_n|}_{\text{converges. (comparison test)}}$$

then $$\underbrace{\sum_{n=1}^{\infty} a_n + |a_n| - |a_n|}_{\text{converges}} \Leftarrow \underbrace{\sum_{n=1}^{\infty} a_n + |a_n|}_{\text{converges}} - \underbrace{\sum_{n=1}^{\infty} |a_n|}_{\text{converges}}$$

$$= \sum_{n=1}^{\infty} a_n$$

$\Rightarrow$ converges. $\square$

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ not absolutely convergent.

Q: $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ abs. convergent? **A:** apply integral test to $\sum \frac{1}{n\ln(n)}$

no.

Q does $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$ converge?

Def^n conditional convergence $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ does not converge.

Th^m Alternating series test

Let $\{a_n\}$ be decreasing, positive sequence, $a_n \to 0$

Then $\underbrace{\sum_{n=1}^{\infty} (-1)^n a_n}_{= s}$ converges. Furthermore: $0 \leq s \leq a_1$

$s_{2n} \leq s \leq s_{2n+1}$ for all n.

Proof even partial sums: $S_{2n} = \underbrace{a_1 - a_2}_{>0} + \underbrace{a_3 - a_4}_{>0} + \cdots + \underbrace{a_{2n-1} - a_{2n}}_{>0}$

~~positive~~ increasing sequence

odd partial sums: $S_{2n+1} = a_1 - \underbrace{(a_2 - a_3)}_{>0} - \underbrace{(a_4 - a_5)}_{>0} - \cdots - \underbrace{(a_{2n} - a_{2n-1})}_{>0}$

decreasing sequence

furthermore $S_{2n} = a_1 - (a_2 - a_3) - \cdots - a_{2n}$

so $S_{2n} \leq a_1$ for all n

so S_{2n} is an increasing sequence bounded above

so $\lim\limits_{n \to \infty} S_{2n}$ exists

similarly $\lim\limits_{n \to \infty} S_{2n+1}$ exists

but $\lim\limits_{n \to \infty} S_{2n} - S_{2n+1} = \lim\limits_{n \to \infty} S_{2n} - \lim\limits_{n \to \infty} S_{2n+1}$

$\qquad\qquad\quad \|$

$\lim\limits_{n \to \infty} -a_{2n+1} = 0$ □.

Example show $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ converges (alternating harmonic series)

use alternating series test: $a_n = \frac{1}{n}$

then a_n positive decreasing series and $a_n \to 0 \Rightarrow \sum\limits_{n=1}^{\infty} (-1)^n a_n$ converges. □.

so $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ is conditionally convergent, but not absolutely convergent

§10.5 Ratio and root tests

fact: $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ Q: how do we show this converges?

(A: comparison test $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-2)(n-1)n > (n-1)^2$ so

$\frac{1}{n!} < \frac{1}{(n-1)^2}$.)

<u>Thm</u> <u>Ratio test</u> $\{a_n\}$ sequence, and suppose $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists. Then ① if $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

② if $\rho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

③ if $\rho = 1$ no information.

<u>Proof</u> if $\rho < 1$, then there is a number $\rho < r < 1$, and as

$\left|\frac{a_{n+1}}{a_n}\right| \to \rho$, there is an M s.t. $\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq M$.

so
$$|a_{M+1}| < r\,|a_M|$$
$$|a_{M+2}| < r\,|a_{M+1}| < r^2\,|a_M|$$
etc.

so $\sum_{n=M}^{\infty} |a_n| \leq \sum_{n=0}^{\infty} |a_M| r^n \leq \frac{|a_M|}{1-r}$ so converges by comparison test.

if $\rho > 1$, then choose r s.t. $\rho > r > 1$,
then as $\left|\frac{a_{n+1}}{a_n}\right| \to \rho$, there is an M s.t. $\left|\frac{a_{n+1}}{a_n}\right| > r > 1$ for all $n \geq M$

then
$$|a_{M+1}| > r\,|a_M|$$
$$|a_{M+2}| > r^2\,|a_M|$$
etc.

so $\sum_{n=M}^{\infty} |a_n| \geq |a_M| \sum_{n=1}^{\infty} r^n$ diverges. □

<u>Example</u> show $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

ratio test $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left|\frac{1/(n+1)!}{1/n!}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$