<u>Def^n</u> The N-th partial sum $S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$

<u>Def^n</u> The sum of the infinite series is defined to be the limit of the partial sums, <u>if this limit exists.</u> $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$

If $\lim_{N\to\infty} S_N = s$ then we say $\sum_{n=1}^{\infty} a_n$ <u>converges</u> and write $\sum_{n=1}^{\infty} a_n = s$.

<u>Example</u>

① $1+1+1+1+\cdots$ $\quad S_N = \underbrace{1+1+\cdots+1}_{N \text{ times}} = N \quad \lim_{N\to\infty} N = \infty$

so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1$ does <u>not</u> converge.

② $1-1+1-1+\cdots$ $\quad s_1 = 1 \quad s_2 = 0 \quad s_3 = 1 \quad s_4 = 0, \ldots$

$s_n = \frac{1-(-1)^n}{2} \qquad (s_n) = 1, 0, 1, 0, 1, \ldots$ does <u>not</u> converge.

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aside

$(1-1)+(1-1)+\cdots = 0+0+0\cdots = 0.$

$1-(1-1)-(1-1)+\cdots = 1-0-0-0\cdots = 1.$

<u>Geometric series</u>

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad a_n = \frac{1}{2^n}$$

$s_1 = \frac{1}{2}$

$s_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1-\frac{1}{4}$

$s_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} = 1-\frac{1}{8}$

$s_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16} = 1-\frac{1}{16}$

$$S_N = \frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^N}$$

$$\frac{1}{2}S_N = \frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{N+1}}$$

$$S_N - \frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$\frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$S_N = 1-\frac{1}{2^N}$$

$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1-\frac{1}{2^N} = 1$, so $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1$.

<u>General case</u> $c + cr + cr^2 + cr^3 + \cdots = \sum_{n=0}^{\infty} cr^n$

$S_N = c + cr + cr^2 + \cdots + cr^N$

$rS_N = cr + cr^2 + cr^3 + \cdots + cr^{N+1}$

$S_N - rS_N = c - cr^{N+1}$

$S_N(1-r) = c(1-r^{N+1})$

$S_N = \dfrac{c(1-r^{N+1})}{1-r}$ $\quad \lim_{N\to\infty} S_N = \lim_{N\to\infty} \dfrac{c(1-r^{N+1})}{1-r} = \dfrac{c}{1-r}$ if $|r|<1$

Otherwise doesn't converge.

<u>Special case</u> (Telescoping)

$S = \sum_{n=1}^{\infty} \dfrac{1}{n(n+1)} = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \cdots$

$\dfrac{1}{n(n+1)} = \dfrac{1}{n} + \dfrac{-1}{n+1}$ $\qquad S_1 = 1 - \dfrac{1}{2}$.

$S_2 = \dfrac{1}{1} - \dfrac{1}{2} + \dfrac{1}{2} - \dfrac{1}{3} = 1 - \dfrac{1}{3}$.

$S_3 = \dfrac{1}{1} - \dfrac{1}{2} + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{3} - \dfrac{1}{4} = 1 - \dfrac{1}{4}$

so $S_N = 1 - \dfrac{1}{N+1}$ $\quad \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \dfrac{1}{N+1} = 1$.

<u>Example</u>

$\underbrace{1}_{\geq 1} + \underbrace{\tfrac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\tfrac{1}{3} + \tfrac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}}_{\geq 1/2} + \underbrace{\tfrac{1}{9} + \cdots + \tfrac{1}{16}}_{\geq \frac{1}{2}} + \cdots$ (harmonic series) (diverges).

<u>Q</u>: when does a series converge? A: dunno.

Tools: Thm Divergence test: $\sum_{n=1}^{\infty} a_n$ If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Warning If $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

Recall: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$ diverges.

Rules for infinite sums / AKA series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series

then
$$\sum a_n + \sum b_n = \sum (a_n+b_n)$$
$$\sum a_n - \sum b_n = \sum (a_n-b_n)$$
$$\sum c a_n = c \sum a_n$$

observation
$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2}^{2n}}$$

§10.3 Series with positive terms

positive series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ all $a_n \geq 0$.

note: the sequence S_N is an increasing sequence
$$S_{N+1} = S_N + \underbrace{a_{N+1}}_{\geq 0}$$

Recall an increasing sequence with an upper bound converges.

Thm Let $S = \sum_{n=1}^{\infty} a_n$ be a positive series. Then exactly one of the following occurs:

① the partial sums are bounded above and S converges

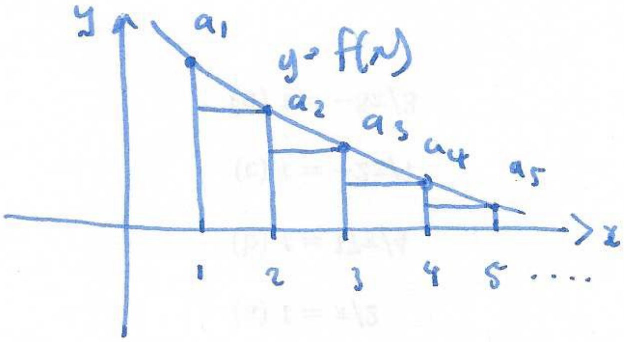
② the partial sums are not bounded above and S diverges.

<u>Thm</u> <u>Integral test</u>. Let $a_n = f(n)$ $f(x)$ positive, decreasing, continuous for $x \geq 1$

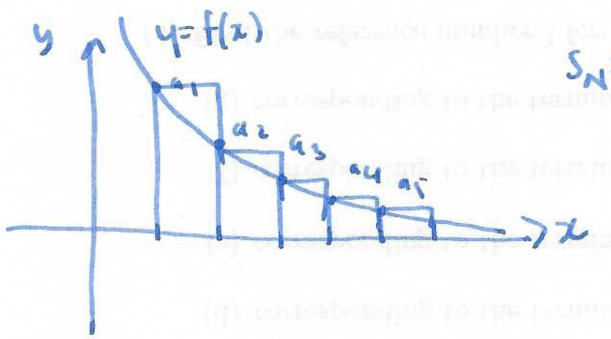
then ① if $\int_1^\infty f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

② if $\int_1^\infty f(x)\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

<u>Proof</u>



$$S_N - a_1 = a_2 + a_3 + \cdots + a_N \leq \int_1^N f(x)\,dx$$



$$S_N = a_1 + a_2 + \cdots + a_N \geq \int_1^N f(x)\,dx.$$

$\square$.

<u>Example</u> $\quad S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \qquad f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$$\int_1^\infty x^{-1/2}\,dx = \lim_{N\to\infty} \int_1^N x^{-1/2}\,dx = \lim_{N\to\infty} \left[2x^{1/2}\right]_1^N = \lim_{N\to\infty} 2\sqrt{N} - 2 \to \infty$$

as $N \to \infty$.

so S diverges.

<u>Thm</u> Convergence of p-series $\quad \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$

<u>Proof</u> (integral test) $\quad \sum_{n=1}^{\infty} a_n \quad a_n = f(n), \; f(x) = \frac{1}{x^p} = x^{-p}$

converges if $p > 1$
diverges if $p \leq 1$. $\square$.

Example $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges $\left(=\frac{\pi^2}{6}\right)$.

$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\dots = \sum_{n=1}^{\infty}\frac{1}{n^3}$ converges ?

Thm Comparison test suppose $0 \leq a_n \leq b_n$ for all $n \geq M$

then ① if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

② if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example $\sum_{n=1}^{\infty} 2^{-n^2} = 2^{-1}+2^{-4}+2^{-9}+\dots$

$= \frac{1}{2}+\frac{1}{2^4}+\frac{1}{2^9}+\dots$

note $\frac{1}{2^{n^2}} < \frac{1}{2^n}$ geometric series, converges.

Thm Limit comparison test a_n, b_n positive sequences

suppose ~~lim~~ $\lim_{n\to\infty}\frac{a_n}{b_n} = L < \infty$ exists.

Then: • if $L>0$ $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.

• if $L=0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof (sketch)

case $L>0$ then $\frac{a_n}{b_n} \to L$ so $0 < \frac{a_n}{b_n} < R$ $L<R$

$0 < a_n < Rb_n$

comparison test: b_n converges $\Rightarrow$ a_n converges