Example $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for any R.

proof there is an integer M s.t. $M \leq R \leq M+1$

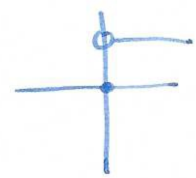
$$0 \leq \frac{R^n}{n!} = \underbrace{\frac{R}{1}\cdot\frac{R}{2}\cdots\cdots\frac{R}{M}}_{\text{call this } C}\cdot\underbrace{\frac{R}{M+1}\cdots\cdots\frac{R}{n-1}}_{\leq 1}\frac{R}{n} \leq C\frac{R}{n}$$

but $\lim_{n\to\infty} C\frac{R}{n} = 0$ so $\lim_{n\to\infty}\frac{R^n}{n!} = 0$.

Thm If $f(x)$ is continuous and $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$$

Important: f continuous. Bad example $f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0\end{cases}$



then $\frac{1}{n} \to 0$

$f(\frac{1}{n}) = 1$ for all n but $f(0) = 0 \neq 1$

Example find $\lim_{n\to\infty} e^{n/n+1}$ start with $\lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+1/n} = 1$

so $\lim_{n\to\infty} e^{n/n+1} = e^{\lim_{n\to\infty} n/n+1} = e^1 = e$.

Defn A sequence $\{a_n\}$ is

- bounded above if $a_n \leq M$ for all n
- bounded below if $L \leq a_n$ for all n
- bounded if $L \leq a_n \leq M$ for all n.

Thm Convergent ~~sub~~sequences are bounded.

Warning bounded sequences need not converge.

Example $a_n = 1, 0, 1, 0, 1, 0, \dots$ $\qquad a_n = \frac{1-(-1)^n}{2}$

Thm Bounded monotonic sequences converge

- if $\{a_n\}$ is increasing and $a_n \leq M$ then $a_n \to L \leq M$.
- if $\{a_n\}$ is decreasing and $a_n \geq \cancel{m}$ then $a_n \to L \geq \cancel{m}$

Example $a_n = \frac{1}{n}$ decreasing : want $\frac{1}{n} > \frac{1}{n+1}$

$n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$.

lower bound $L = -100$

so $\lim_{n \to \infty} \frac{1}{n} \geq -100$

Example show $a_n = \sqrt{n+1} - \sqrt{n}$ decreasing and bounded below.

Try: $f(x) = \sqrt{x+1} - \sqrt{x} = (x+1)^{1/2} - (x)^{1/2} > 0$ as $\sqrt{x}$ monotonic.
so boundeded below by zero.

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right)$$

claim : $f'(x) < 0$: $\quad x+1 > x$

$\sqrt{x+1} > \sqrt{x}$

$\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}}$. $\quad$ so $f'(x) < 0$.
$\Rightarrow$ decreasing.

§10.2 Series

Defn A series is an infinite sum $a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$

Examples $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ $\qquad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

$1 + 1 + 1 + 1 + \cdots$ $\qquad 1 - 1 + 1 - 1 + 1 - 1 + \cdots$