Q: how good is the approximation?

Thm (Error bound) $f(x)$ function, $f^{(n)}(x)$ exists and is continuous.

Let K be an upper bound on $|f^{(n+1)}(u)|$ for all u in $[a,x]$

then $$\left|T_n(x) - f(x)\right| \le \frac{K|x-a|^{n+1}}{(n+1)!}$$

Example $f(x) = e^x$ find $T_4(x)$ at $x=0$ find an error bound for $T_4(1)$.

$f(x) = e^x \quad f(0) = 1$

$f'(x) = e^x \quad f'(0) = 1$

$f''(x) = e^x \quad f''(0) = 1$

$$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$T_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx 2.708\overline{33}\ldots$$

error bound: need to find K s.t. $|f^{(n+1)}(u)| \le K$ for $u \in [0,1]$.

i.e. $|e^u| \le K$ for $u \in [0,1]$; can choose $K = e$.

so $$\left|e - T_4(1)\right| \le \frac{e \cdot 1^{n+1}}{(n+1)!} = \frac{e}{120}$$

Example $f(x) = \frac{1}{1-x}$

$f(x) = \frac{1}{1-x} = (1-x)^{-1} \quad f(0) = 1$

$f'(x) = +(1-x)^{-2} \quad f'(0) = 1$

$f''(x) = 2(1-x)^{-3} \quad f''(0) = 2!$

$f^{(3)}(x) = 2.3(1-x)^{-4} \quad f^{(3)}(0) = 3!$

$$T_n(x) = 1 + 1.x + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \cdots$$

$$= 1 + x + x^2 + x^3 + \cdots$$

§10.1 Sequences

A sequence is a list of numbers, indexed by $\mathbb{N}$ = positive integers.

Examples $1, 2, 3, 4, \ldots$ **Notation** $a_1, a_2, a_3, \ldots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ or $\{a_n\}_{n \in \mathbb{N}}$

$1, 27, \pi, \sqrt{2}, \ldots$

a_n is the n-th number in the sequence.

Sometimes we can give the sequence by a formula (depending on n)

Examples $\{a_n\} = \{n\} = 1, 2, 3, 4, \ldots$

$\{a_n\} = \{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$

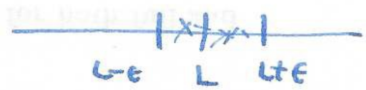
but there doesn't need to be a formula.

Example (recursive def$^{\underline{n}}$) $a_{n+2} = a_n + a_{n+1}$ $a_1 = a_2 = 1$.

gives $1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci sequence)

sequences may converge.

Def$^{\underline{n}}$ A sequence $\{a_n\}$ converges to L if for every $\epsilon > 0$ there is an N s.t. $|a_n - L| \leq \epsilon$ for all $n \geq N$.



Notation $\lim_{n \to \infty} a_n = L$ or $a_n \to L$

Examples $\{a_n\} = \{\frac{1}{n}\}$ $\lim_{n \to \infty} \frac{1}{n} = 0$

$a_n = \frac{1}{n}$

check: pick $\epsilon > 0$, then choose $N > \frac{1}{\epsilon}$ if $n > N$ then $\frac{1}{n} < \frac{1}{N} < \epsilon$.

special case: sequence defined by a function $f(x)$, i.e. $a_n = f(n)$

Th$^{\underline{m}}$ If $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} f(n) = L$ Q: is converse true?

Example $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ $a_n = \frac{n-1}{n}$ $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$

$\lim_{x \to \infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim_{n \to \infty} \frac{n-1}{n} = 1.$

Example Geometric sequences $a_n = r^n$

eg $2, 4, 8, 16, 32, \ldots$ $a_n = 2^n$

$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$ $a_n = \frac{1}{3^n}$

$1, 1, 1, 1, \ldots$ $a_n = 1^n$

Fact $\lim_{n \to \infty} r^n = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & |r| < 1 \end{cases}$

Rules for limits of sequences: same as rules for limits of functions.

Assume $a_n \to L$, $b_n \to M$, then

- $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = L + M$
- $\lim_{n \to \infty} a_n b_n = \left(\lim_{n \to \infty} a_n\right)\left(\lim_{n \to \infty} b_n\right) = LM.$
- $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\left(\lim_{n \to \infty} a_n\right)}{\left(\lim_{n \to \infty} b_n\right)} = \frac{L}{M}$
- $\lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n = cL$ (c constant)

Squeeze thm if $a_n \leq b_n \leq c_n$ and $\lim_{n \to \infty} a_n = L$, $\lim_{n \to \infty} c_n = L$

then $\lim_{n \to \infty} b_n = L$