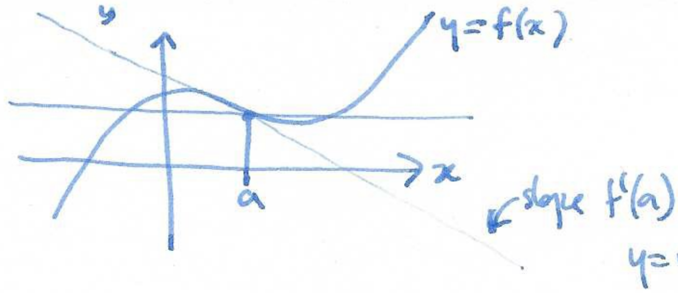
§8.4 Taylor polynomials

Approximating functions:



$y = mx + c$

$y - y_0 = m(x - x_0)$

0-th approx: $f(x) \approx f(a)$

1st approx (straight line) $f(x) \approx f(a) + f'(a)(x-a)$ ⟵ $y - f(a) = f'(a)(x-a)$

2nd approx (quadratic) } how do we find these?

3rd approx (cubic) }

⋮

- choose quadratic whose first two derivatives agree with $f(x)$ at $x=a$.
- choose cubic whose first 3 derivatives agree with $f(x)$ at $x=a$.
- and so on...

quadratic $\quad T_2(x) = a_0 + a_1(x-a) + a_2(x-a)^2$

deg 0 : $\quad T_2(a) = a_0 + 0 + 0 = f(a)$ so $a_0 = f(a)$

deg 1 : $\quad T_2'(x) = a_1 + 2a_2(x-a)$

$\quad T_2'(a) = a_1 + 0 = f'(a)$ so $a_1 = f'(a)$

deg 2 : $\quad T_2''(x) = 2a_2 = f''(a)$ so $a_2 = \frac{1}{2} f''(a)$

so : $\quad T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

in general $\quad T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$

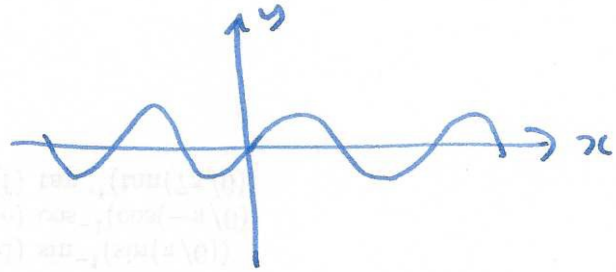
differentiate k times

$$T_n^{(k)}(x) = k!\,a_k + (\text{stuff with } (x-a) \text{ factors}).$$

$$T_n^{(k)}(a) = k!\,a_k = f^{(k)}(a) \quad \text{so} \quad a_k = \frac{1}{k!} f^{(k)}(a).$$

or $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{n!}(x-a)^k$

Examples ① $y = \sin(x)$ at $x = 0$



$f(x) = \sin(x) \quad f(0) = 0$

$f'(x) = \cos(x) \quad f'(0) = 1$ so

$f''(x) = -\sin(x) \quad f''(0) = 0$

$f^{(3)}(x) = -\cos(x) \quad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin(x) \quad f^{(4)}(0) = 0$

$T_n(x) = 0 + 1\cdot x + 0\cdot\frac{x^2}{2!} + (-1)\frac{x^3}{3!} + 0\cdot\frac{x^4}{4!} + 1\cdot\frac{x^5}{5!} + \cdots$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

② $y = \cos(x)$ at $x = 0$

$f(x) = \cos(x) \quad f(0) = 1$

$f'(x) = -\sin(x) \quad f'(0) = 0$

$f''(x) = -\cos(x) \quad f''(0) = -1$

$f^{(3)}(x) = \sin(x) \quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x) \quad f^{(4)}(0) = 1$

$T_n(x) = 1 + 0x + (-1)\frac{x^2}{2!} + 0\frac{x^3}{3!} + (1)\frac{x^5}{5!} + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

③ $y = e^x$ at $x = 0$

$f(x) = e^x \quad f(0) = 1$

$f'(x) = e^x \quad f(0) = 1$

$f''(x) = e^x \quad f(0) = 1$

$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

note $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$

$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$

$$e^{i\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos\theta + i\sin\theta$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \qquad e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

$$e^{i2\theta} = \cos 2\theta + i\sin 2\theta$$

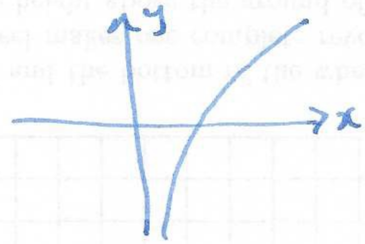
$$\parallel$$

$$(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

<u>Example</u> $y = \ln(x)$ at $x = 1$



$f(x) = \ln(x)$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\qquad f'(1) = 1$

$f''(x) = -x^{-2}$ $\qquad f''(1) = -1$

$f^{(3)}(x) = +2x^{-3}$ $\qquad f^{(3)}(1) = 2$

$f^{(4)}(x) = \underset{2.3}{-6}\,x^{-4}$ $\qquad f^{(4)}(1) = -2.3$

$f^{(5)}(x) = 2.3.4.\,x^{-5}$ $\qquad f^{(5)}(1) = 4!$

$f^{(k)}(x) = (k-1)!\,(-1)^{k+1}x^{-k}$ $\qquad f^{(k)}(1) = (-1)^{k+1}(k-1)!$

so $$T_n(x) = 0 + 1(x-1) + (-1)\frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} + (-3!)\frac{(x-1)^4}{4!} + \cdots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

<u>note</u>: what is $T_n(3)$? $\quad 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \cdots$ doesn't converge.