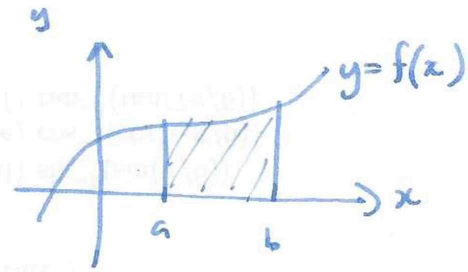
note $\int \frac{1}{1+x^2}dx = \tan^{-1}(x)+c$ $\qquad \int \frac{x}{1+x^2}dx = \frac{1}{2}\ln|1+x^2|+c$

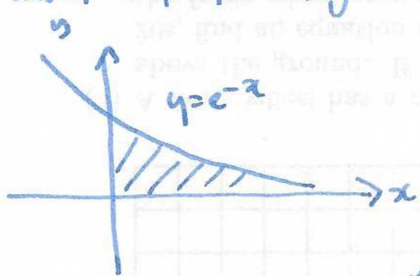
7.6 Improper integrals



recall $\int_a^b f(x)\,dx$ = area under the curve

Q: what about infinite integrals?

Example



$\int_0^\infty e^{-x}dx$ note: $\int_0^R e^{-x}dx = \left[-e^{-x}\right]_0^R$

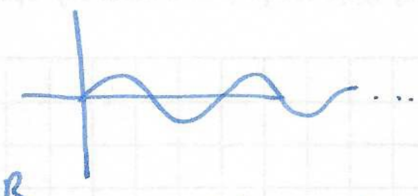
$= -e^{-R}+e^0 = 1-e^{-R}$

Def^n $\int_a^\infty f(x)\,dx = \lim_{R\to\infty}\int_a^R f(x)\,dx$, if this limit exists.

Example $\int_0^\infty e^{-x}dx = \lim_{R\to\infty} 1-e^{-R} = 1$

Warning: sometimes the limit doesn't exist.

Example $\int_0^\infty \sin(x)\,dx$

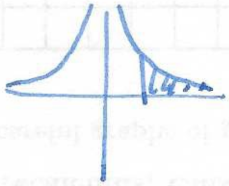


$\int_0^R \sin(x)\,dx = \left[-\cos(x)\right]_0^R = -\cos(R)+\cos(0) = 1-\cos(R)$

$\lim_{R\to\infty} 1-\cos(R)$ DNE! e.g. $R = 2\pi N$: $1-\cos(R) = 0$

$2\pi N+\frac{\pi}{2}$: $1-\cos(R) = 1$

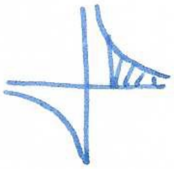
Example $\int_1^\infty \frac{1}{x^2}dx$



$\lim_{R\to\infty}\int_1^R \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^R = -\frac{1}{R}+1$

$\lim_{R\to\infty} 1-\frac{1}{R} = 1$.

Example $\int_1^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty} \left[\ln|x|\right]_1^R = \ln|R| - \ln|1|$



$\lim_{R\to\infty} \ln|R| = \infty.$

Doubly infinite integrals $\int_{-\infty}^{\infty} f(x)\,dx$

(f cts!)

Defn $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx$ ← provided each limit exists!

Example $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$

$$= \lim_{R\to\infty} \int_{-R}^{0} \frac{1}{1+x^2}\,dx + \lim_{R\to\infty} \int_0^R \frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to\infty} \left[\tan^{-1}(x)\right]_{-R}^{0} + \lim_{R\to\infty} \left[\tan^{-1}(x)\right]_0^R$$

$$= \lim_{R\to\infty} \left(0 - \tan^{-1}(-R)\right) + \lim_{R\to\infty} \left(\tan^{-1}(R) - 0\right)$$

$$= \lim_{R\to\infty} \tan^{-1}(R) + \lim_{R\to\infty} \tan^{-1}(R) = \pi.$$

Warning: $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$

Example $\underset{\text{DNE.}}{\int_{-\infty}^{\infty} \sin(x)\,dx} \neq \lim_{R\to\infty} \int_{-R}^{R} \sin(x)\,dx = \lim_{R\to\infty} \left[\cos x\right]_{-R}^{R} = \lim_{R\to\infty} 0 = 0.$

Example $\int_0^\infty x e^{-x}\,dx = \lim_{R\to\infty} \int_0^R \underbrace{x}_{u}\underbrace{e^{-x}}_{v'}\,dx$ $\qquad \int uv'\,dx = uv - \int u'v\,dx$

$u = x \quad u' = 1$

$v' = e^{-x} \quad v = -e^{-x}$

$= \lim_{R\to\infty} \left[-xe^{-x}\right]_0^R + \int_0^R e^{-x}\,dx = \lim_{R\to\infty} -Re^{-R} + \left[-e^{-x}\right]_0^R = \lim_{R\to\infty} e^{-R}(-R-1) + 1$