§7.6 Partial fractions

aim: $\int \frac{P(x)}{Q(x)}\, dx \qquad P(x), Q(x)$ polynomials

recall: $\int \frac{1}{x+a}\, dx = \ln|x+a| + c$

a_i all distinct real numbers.

special case: $\deg(P) < \deg(Q)$ and $Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

then $\frac{P(x)}{Q(x)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \cdots + \frac{A_n}{x+a_n} \qquad A_i$ numbers.

Example $\int \frac{x}{x^2-1}\, dx \qquad x^2-1 = (x-1)(x+1)$

$$\frac{x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$\frac{x}{x^2-1} = \frac{x(A+B)+(B-A)}{(x+1)(x-1)}$$

$$x = 1x + 0 = (A+B)x + (B-A)$$

$\left.\begin{array}{l} ① \ A+B=1 \\ ② \ B-A=0 \end{array}\right\}$ ①+②: $2B = 1 \quad B = 1/2, \ A = 1/2$

[short cut: works for all $x \Rightarrow$ works for $x = 1, -1$].

$x = 1: \quad 1 = 2B \quad , \quad x = -1: \quad -1 = -2A$

$$\int \frac{x}{x^2-1}\, dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1}\, dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c.$$

special case: suppose $\deg P > \deg Q$? long division

Example $\int \frac{x^3}{x^2-1}\, dx$

$$\begin{array}{r|l} & x \\ x^2-1 & x^3 \\ & \underline{x^3 - x} \\ & x \end{array}$$

so $x^3 = (x^2-1)x + x$.

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1} = x + \frac{1/2}{x+1} + \frac{1/2}{x-1}$$

So $\displaystyle\int \frac{x^3}{x^2-1}\,dx = \int x + \frac{1/2}{x+1} + \frac{1/2}{x-1}\,dx = \frac{1}{2}x^2 + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c$

<u>special case</u> $\dfrac{P(x)}{Q(x)}$ $\quad Q(x) = (x+a_1)\cdots(x+a_n)$

a_i <u>not</u> all distinct, i.e. repeated factors.

<u>Example</u> $\displaystyle\frac{x+1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ $\quad A, B, C$ numbers.

<u>In general</u> $\dfrac{P(x)}{Q(x)}$, $Q(x) = (x+a_1)^{n_1}(x+a_2)^{n_2}\cdots(x+a_k)^{n_k}$

then $$\frac{P(x)}{Q(x)} = \frac{A_{1,1}}{x+a_1} + \frac{A_{1,2}}{(x+a_1)^2} + \dots + \frac{A_{1,n_1}}{(x+a_1)^{n_1}} + \dots + \frac{A_{k,1}}{x+a_k} + \frac{A_{k,2}}{(x+a_k)^2} + \dots + \frac{A_{k,n_k}}{(x+a_k)^{n_k}}.$$

<u>Example</u> $$\frac{x+1}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$$

equate coefficients: 3 equations in 3 unknowns.

true for all $x \Rightarrow$ true for any particular x:

$x=-2$: $\quad -1 = C(-3) \Rightarrow C = \frac{1}{3}$.

$x=1$: $\quad 2 = A(9) \Rightarrow A = \cancel{4}\frac{2}{9}$.

$x=0$: $\quad 1 = 4A + -2B - C \Rightarrow 2B = 4A - C - 1 = \frac{8}{9} - \frac{1}{3} - 1 = \frac{-4}{9}$.

Quadratic factors

$$x^2+1 = (x-i)(x+i) \neq (x+a_1)(x+a_2) \quad a_1, a_2 \text{ real}.$$

complex roots!

note: $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$

Q: $\displaystyle\int \frac{1}{1+x+x^2}\,dx$? A: complete the square: $x^2+x+1 = (x+\tfrac{1}{2})^2 + \tfrac{3}{4}$

$$= \int \frac{1}{3/4 + (x+1/2)^2}\,dx \qquad \text{sub } u = x+\tfrac{1}{2} \quad \frac{du}{dx} = 1 \qquad \int \frac{1}{3/4+u^2}\,du \quad \text{etc...}$$

linear and quadratic factors

$$\int \frac{1}{x(1+x^2)}\,dx \qquad \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$0x^2+0x+1 = x^2(A+B) + x(C) + A \qquad \left.\begin{array}{l} A+B=0 \\ C=0 \\ A=1 \end{array}\right\} \Rightarrow B=-1.$$

$$\int \frac{1}{x} + \frac{-x}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + c.$$

repeated quadratic factors

$$\frac{1}{(x+a)(x+b)^2(x^2+c)^2} = \frac{A}{x+a} + \frac{B}{(x+b)} + \frac{C}{(x+b)^2} + \frac{Dx+E}{x^2+c} + \frac{Fx+G}{(x^2+c)^2}$$

solve for numbers $A \ldots G$.