a odd:
$$\int \tan^3 x \sec^2 x \, dx = \int \tan^2 x \sec x (\tan x \sec x) \, dx$$

$$= \int (1-\sec^2 x) \sec x (\tan x \sec x) \, dx \qquad u = \sec x \qquad \frac{du}{dx} = \sec x \tan x$$

$$= \int (1-u^2) u \frac{\tan x \sec x}{\tan x \sec x} \, du = \int u - u^3 \, du = \frac{1}{2} u^2 - \frac{1}{4} u^4 + c$$

$$= \frac{1}{2} \sec^2 x - \frac{1}{4} \sec^4 x + c.$$

b even:
$$\int \tan^3 x \sec^2 x \, dx \qquad u = \tan x \qquad \frac{du}{dx} = \sec^2 x$$

$$= \int u^3 \frac{\sec^2 x}{\sec^2 x} \, du = \frac{1}{4} u^4 + c = \frac{1}{4} \tan^4 x + c$$

a even, b odd: write as powers of $\sec(x)$ and use integration by parts.

__Example__ $\int \sin 3x \cos 2x \, dx$? useful fact: $\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$.

$$\left.\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \sin(A-B) &= \sin A \cos B - \cos A \sin B \end{aligned}\right\} \qquad \sin(A+B) + \sin(A-B) = 2 \sin A \cos B.$$

$$\int \sin 3x \cos 2x \, dx = \int \frac{1}{2} \big(\sin(3x+2x) + \sin(3x-2x) \big) dx$$

$$= \frac{1}{2} \int \sin 5x + \sin x \, dx = \frac{1}{2} \left(-\frac{1}{5} \cos 5x - \cos x \right) + c.$$

__Example__ $\int \cos 4x \cos 7x \, dx$?
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B.$$

$$= \frac{1}{2} \int \cos(4x+7x) + \cos(4x-7x) \, dx.$$

§7.3 Trig substitutions

aim: deal with $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$
(1), (2), (3)

use:

$$\left.\begin{array}{l} \cos^2 x + \sin^2 x = 1 \quad \leftrightarrow \quad \sin^2 x = 1 - \cos^2 x \;\; (1) \\ \leftrightarrow \cot^2 x + 1 = \operatorname{cosec}^2 x \quad \leftrightarrow \quad \cot^2 x = \operatorname{cosec}^2 x - 1 \;\; (3) \\ \leftrightarrow 1 + \tan^2 x = \sec^2 x \;\; (2) \quad \cancel{\leftrightarrow \tan^2 x = \sec^2 x} \end{array}\right\} \text{to convert } \sqrt{-} \text{ into } \sqrt{\text{perfect square}}.$$

<u>Example</u> $\int \sqrt{9-x^2}\,dx$ try: $x = 3\sin u$

$$\frac{dx}{du} = 3\cos u$$

$$\int \sqrt{9-(3\sin u)^2}\,\frac{dx}{du}\,du$$

$$\int \sqrt{9 - 9\sin^2 u}\; 3\cos u\,du = 3\int \sqrt{\underbrace{1-\sin^2 u}_{\cos^2 u}}\cdot 3\cos u\,du$$

$$= 9\int \cos^2 u\,du \quad \text{etc.}$$

<u>Example</u> $\int \sqrt{1+4x^2}\,dx$ try $x = \frac{1}{2}\tan u$

$$\frac{dx}{du} = \frac{1}{2}\sec^2 u$$

$$\int \sqrt{1+4\left(\tfrac{1}{2}\tan u\right)^2}\,dx = \int \sqrt{\underbrace{1+\tan^2 u}_{\sec^2 u}}\,\frac{dx}{du}\,du = \int \sec u \cdot \frac{1}{2}\sec^2 u\,du$$

$$= \frac{1}{2}\int \sec^3 u\,du \ldots \quad (\text{parts} + \text{use } \sec^2 u = 1 + \tan^2 u).$$

<u>Example</u> $\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx$ try $x = a\sec u$

$$\frac{dx}{du} = a\sec u\tan u$$

$$\int \frac{1}{a^2\sec^2 u}\cdot\frac{1}{\sqrt{a^2\sec^2 u - a^2}}\,\frac{dx}{du}\,du = \frac{1}{a^3}\int \frac{1}{\sec^2 u}\,\frac{1}{\tan u}\,a\sec u\tan u\,du = \frac{1}{a^2}\int \cos u\,du$$

$= \frac{1}{a^2} \sin u + c \qquad x = a\sec u = \frac{a}{\cos u} \qquad \cos u = \frac{a}{x} \qquad \sin^2 u + \cos^2 u = 1$

$\sin u = \sqrt{1-\cos^2 u}.$

$$= \frac{1}{a^2}\sqrt{1-\left(\frac{a}{x}\right)^2} + c = \frac{\sqrt{x^2-a^2}}{a^2 x} + c$$

recall aim: $\int \sin^a x \cos^b x \, dx$, $\int \sin(ax)\cos(bx)\, dx \qquad \int \sqrt{\pm a^2 \pm x^2}\, dx$

- trig identities $\cos^2 x + \sin^2 x = 1 \iff \cos^2 x = 1 - \sin^2 x$

  $1 + \tan^2 x = \sec^2 x \iff \tan^2 x = \sec^2 x - 1$

- substitution $x = \sin u, \cos u$ (not $u = \sin x$!)

- parts.

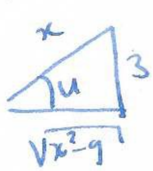
Example $\int \frac{1}{x^2\sqrt{x^2-9}}\, dx \qquad$ try $x = 3\sec u$

$\frac{dx}{du} = 3\sec u \tan u$

$$\int \frac{1}{(3\sec u)^2 \sqrt{9\sec^2 u - 9}} \frac{dx}{du}\, du$$

$$\int \frac{1}{9\sec^2 u \sqrt{9} \underbrace{\sqrt{\sec^2 u - 1}}_{= \sqrt{\tan^2 u}}} 3\sec u \tan u \, du$$

$$\int \frac{1}{9\sec u}\, du = \frac{1}{9}\int \cos u \, du = \frac{1}{9}\sin u + c$$



$x = 3\sec u \qquad \sin(u) = \frac{\sqrt{x^2-9}}{x}$

$$= \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + c \qquad \text{check!}$$