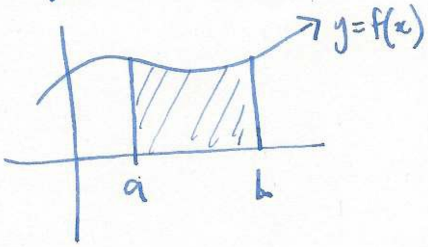
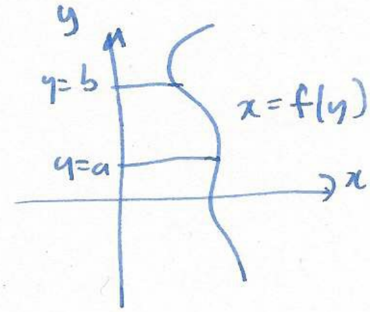
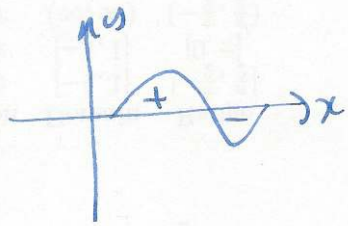
Integration along y-axis

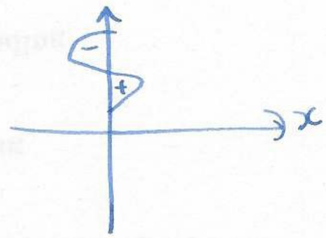
$\int_a^b f(x)\,dx$



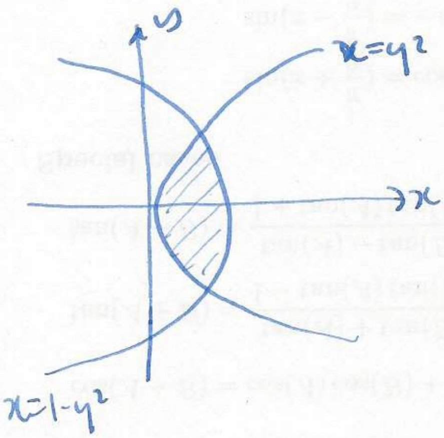
$\int_a^b f(y)\,dy$







<u>Example</u> find area between curves $x=y^2$ and $x=1-y^2$



§6.2 Volume, density, averages

recall: volume of cylinder is Ah

A = area of base.

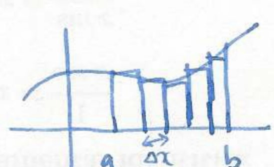
fact: this works for cylinders of any shape:

$V = Ah$.

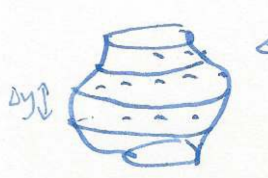
suppose the shape is not a cylinder:

can approximate by horizontal slices

<u>recall</u>:



← can approximate area under curve by rectangles of width Δx.



← can approximate volume of object by cylinders of width Δy say

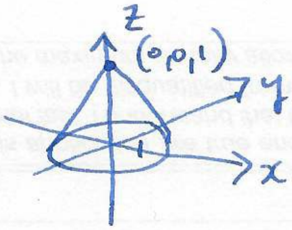
$V = Ah$

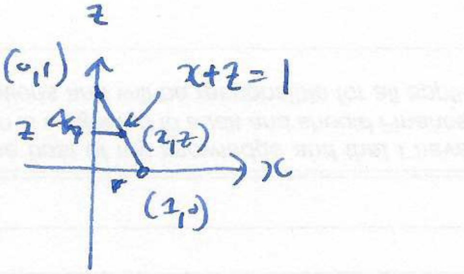
Volume $V = \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} A(y_i)\Delta y$

$$\boxed{\text{Volume} \quad V = \int_a^b A(y)\,dy}$$

Example

Find volume of cone





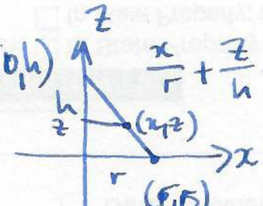
$$V = \int_0^1 A(z)\,dz$$

$A = \pi x^2 = \pi(1-z)^2$.

$$= \int_0^1 \pi(1-z)^2\,dz = \pi\int_0^1 1-2z+z^2\,dz$$

$$= \pi\left[z - z^2 + \frac{1}{3}z^3\right]_0^1 = \pi\frac{1}{3}.$$

in general:

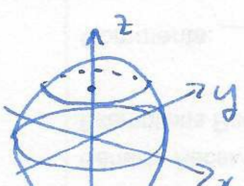


$\frac{x}{r} + \frac{z}{h} = 1$: $x = r\left(1 - \frac{z}{h}\right)$.

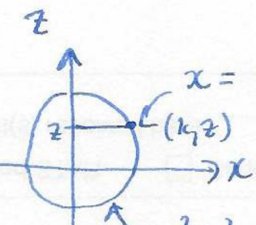
$$V = \int_0^h A(z)\,dz = \int_0^1 \pi r^2\left(1-\frac{z}{h}\right)^2 dz = \pi r^2\int_0^1 1 - \frac{2z}{h} + \frac{z^2}{h^2}\,dz$$

$$= \pi r^2\left[z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right]_0^h = \pi r^2\left(h - \frac{h^2}{h} + \frac{h^3}{3h^2}\right) = \frac{1}{3}\pi r^2 h.$$

Example Volume of sphere



$$\int_{-r}^{r} A(z)\,dz$$

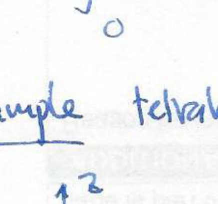


$x = \sqrt{r^2 - z^2}$.

$x^2 + z^2 = r^2$

$$= 2\int_0^r \pi(r^2 - z^2)\,dz = 2\pi\left[r^2 z - \frac{1}{3}z^3\right]_0^r = 2\pi\left(r^3 - \frac{1}{3}r^3\right) = \frac{4}{3}\pi r^3.$$

Example tetrahedron with vertices $(0,0,0)$, $(0,0,1)$ $(0,1,0)$ $(1,0,0)$



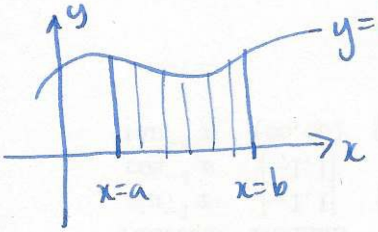
$$V = \int_0^1 A(z)\,dz$$

area of triangle $= \frac{1}{2}$ base × height $= \frac{1}{2}(1-z)(1-z)$

$$= \int_0^1 \frac{1}{2}(1-z)^2\,dz \dots = \frac{1}{6}.$$

<u>recall</u> average of numbers $a_1, a_2, \dots, a_n$ is $\dfrac{a_1 + \cdots + a_n}{n} = \dfrac{1}{n}\sum_{i=1}^{n} a_i$

<u>Q</u>: what about average value of function $f(x)$ on an interval?



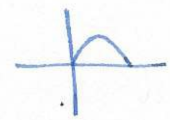
average : $\frac{1}{n}\left(f(x_1) + \cdots + f(x_n)\right)$

<u>recall</u> : $\displaystyle\int_a^b f(x)\,dx \approx \left(f(x_1) + \cdots + f(x_n)\right)\Delta x \qquad \Delta x = \frac{b-a}{n}$

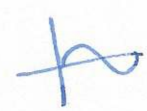
so $\displaystyle\int_a^b f(x)\,dx \approx \text{average} \times (b-a)$ so $\boxed{\text{average} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx}$

<u>Example</u> find average value of $\sin(x)$ on $[0,\pi]$, $[0,2\pi]$

$[0,\pi]$: $\displaystyle\frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{1}{\pi}\Big[-\cos(x)\Big]_0^{\pi} = \frac{1}{\pi}\left(-(-1)+1\right) = \frac{2}{\pi}$

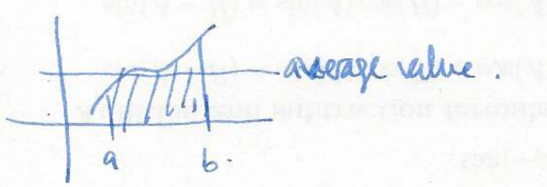


$[0,2\pi]$: $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} \sin(x)\,dx = \frac{1}{2\pi}\Big[-\cos(x)\Big]_0^{2\pi} = \frac{1}{2\pi}\left(-1+1\right) = 0.$



<u>useful fact</u> (mean value theorem for integrals)

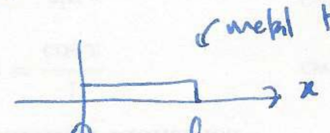
If $f(x)$ is cts on $[a,b]$, then there is a $c \in [a,b]$ s.t. $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$



<u>Example</u> find the average value of $\dfrac{\sin(\pi/x)}{x^2}$ over $[1,2]$

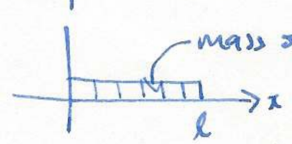
<u>low cunning</u>: which has bigger average on $[0,\pi]$, $\sin(x)$ or $\sin^2 x$?

<u>Density</u>



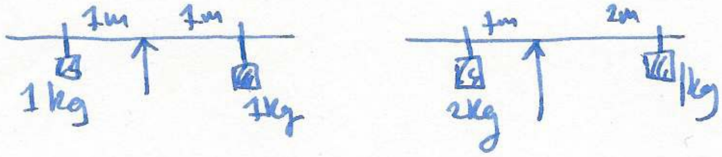
mass $= \displaystyle\int_0^{l} \rho(x)\,dx.$

intuition :

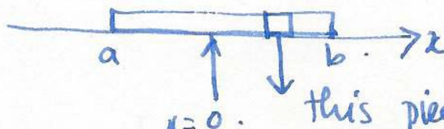


mass of each bit $\approx \rho(x_i)\Delta x$ so total mass $\sum \rho(x_i)\Delta x \approx \displaystyle\int_0^{l} \rho(x)\,dx$

balancing:



moment / turning force about 0:

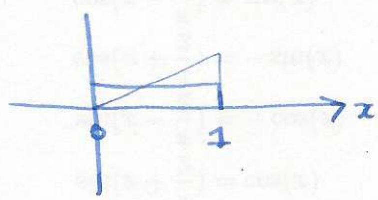


this piece gives turning force of $\underbrace{\rho(x_i)\Delta x}_{\text{mass}} \cdot \underbrace{x_i}_{\text{distance}}$.

total turning force $\sum x_i \rho(x_i) \Delta x$

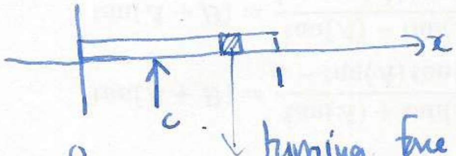
$$\approx \int_a^b x\rho(x)\,dx.$$

Example



$\rho(x) = x$ turning force $= \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$.

center of mass: point c s.t. turning force about c is equal to zero.



turning force of this piece $= \underbrace{\rho(x_i)\Delta x}_{\text{mass}} \cdot \underbrace{(x_i - c)}_{\text{distance}}$

turning force about c: $\sum \rho(x_i)\Delta x_i (x_i - c)$

$$\approx \int_a^b \rho(x)(x-c)\,dx$$

$$= \int_a^b x\rho(x) - c\rho(x)\,dx = \underbrace{\int_a^b x\rho(x)\,dx}_{\text{turning force about zero}} - c\underbrace{\int_a^b \rho(x)\,dx}_{\text{mass}} \quad (*)$$

set $(*) = 0$: $c = \dfrac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx} = \dfrac{1}{\text{mass}} \int_a^b x\rho(x)\,dx.$

Example $\rho(x) = x$ mass is $\int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$.

$c = \dfrac{1/3}{1/2} = 2/3.$