§5.6 Integration by substitution

$$\int_{x=a}^{x=b} f(x)\,dx \quad , \quad x(u) \quad \rightsquigarrow \quad \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u))\frac{dx}{du}\,du$$

recall $\dfrac{dx}{du} = \dfrac{1}{\frac{du}{dx}}$ integration by substitution is "reverse chain rule".

Examples

- $\int x\sin(x^2)\,dx$ try $x^2 = u$ $\dfrac{du}{dx} = 2x$

$$\rightsquigarrow \int x\sin(u)\frac{dx}{du}\,du = \int x\sin(u)\frac{1}{2x}\,du = \int \sin(u)\,du$$

$$= -\cos(u) + C = -\cos(x^2) + C \qquad \text{check!}$$

- $\int e^{4x}\,dx$ $u = 4x$, $\dfrac{du}{dx} = 4$ $\quad \int e^u \frac{1}{4}\,du = \frac{1}{4}e^u + C = \frac{1}{4}e^{4x} + C$ check!

- $\int \tan\theta\,d\theta = \int \dfrac{\sin\theta}{\cos\theta}\,d\theta \quad \left[\text{hint: } \dfrac{d}{dx}\left(\ln(f(x))\right) = \dfrac{1}{f(x)}f'(x)\right]$

try $u = \cos\theta$, $\dfrac{du}{d\theta} = -\sin\theta$ $\quad \int \dfrac{\sin\theta}{u}\cdot\dfrac{1}{-\sin\theta}\,d\theta = -\int \frac{1}{u}\,du = -\ln|u| + C$

$$= -\ln|\cos x| + C \qquad \text{check!}$$

- $\int x\sqrt{x+1}\,dx$ try: $u = x+1$, $\dfrac{du}{dx} = 1$ $\quad \int x\sqrt{u}\,du = \int (u-1)\sqrt{u}\,du$

$$= \int u^{3/2} - u^{1/2}\,du = \frac{2}{5}u^{5/2} - \frac{2u^{3/2}}{3} + C = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C.$$

Definite integrals:

- $\int_0^2 x^2\sqrt{x^3+1}\,dx$ $\quad u = x^3+1$, $\dfrac{du}{dx} = 3x^2$ $\quad \int_1^9 x^2\sqrt{u}\frac{1}{3x^2}\,du = \int_1^9 \frac{1}{3}u^{1/2}\,du = \left[\frac{2}{9}u^{3/2}\right]_1^9$

§5.7 More functions

inverse trig functions

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

$$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + C$$

Example $\displaystyle\int_0^1 \frac{1}{4+x^2}\,dx$ try $x = 2u$ $(x^2 = 4u^2)$ $\quad \frac{dx}{du} = 2 \rightsquigarrow \displaystyle\int_0^{1/2} \frac{1}{4+4u^2}\,2du$

$$= \frac{1}{2}\int_0^{1/2} \frac{1}{1+u^2}\,du = \frac{1}{2}\left[\tan^{-1}(u)\right]_0^{1/2} = \frac{1}{2}\left(\tan^{-1}(1/2) - \tan^{-1}(0)\right) = \frac{1}{2}\tan^{-1}(1/2)$$

exponential functions

$$\frac{d}{dx}(e^x) = e^x \qquad \int e^x\,dx = e^x + C$$

$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{x\ln(b)}\right) = \ln(b)\,e^{x\ln(b)} = \ln(b)\,b^x$$

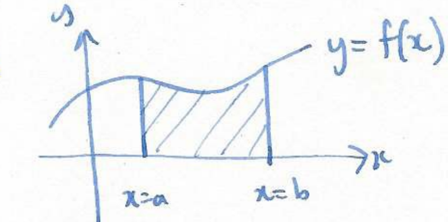
$$\int b^x\,dx = \frac{1}{\ln(b)}\,b^x + C$$

Examples $\displaystyle\int_0^1 10^x\,dx \qquad \int_0^{\pi/2} \cos\theta\, 10^{\sin\theta}\,d\theta$.

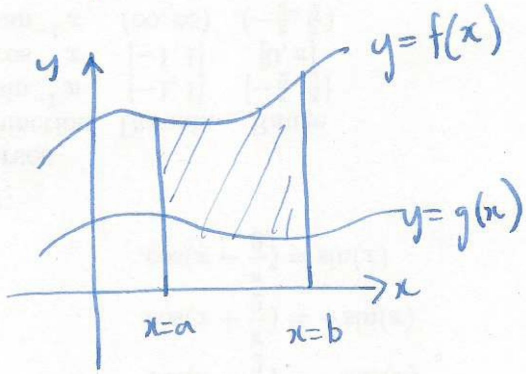
c

§6.1 Area between two curves

recall



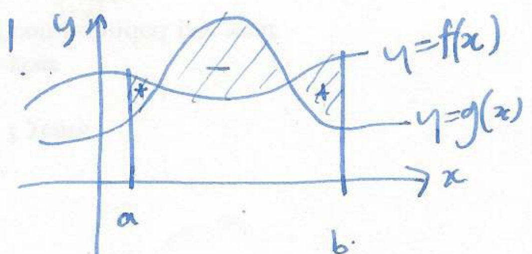
area under graph $\int_a^b f(x)\,dx$



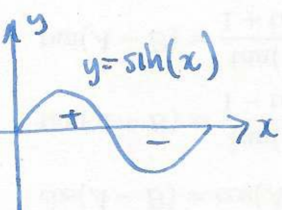
area between two curves

$$= \int_a^b f(x) - g(x)\,dx$$

warning: signs!



Q: what do you do if you want "absolute area"? (unsigned area).

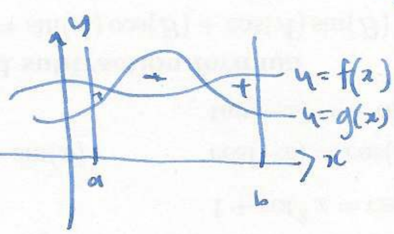


$\int_0^{2\pi} \sin(x)\,dx = 0$ want $\int_0^{2\pi} |\sin(x)|\,dx$ ← need to know when $\sin(x)$ +ve or −ve.

i.e. need to find zeros: $f(x) = 0$. $\int_0^{\pi} \sin(x)\,dx + \int_{\pi}^{2\pi} -\sin(x)\,dx = \left[\cos(x)\right]_0^{\pi} - \left[\cos(x)\right]_{\pi}^{2\pi}$

$= 2 - -2 = 4.$

in this case:

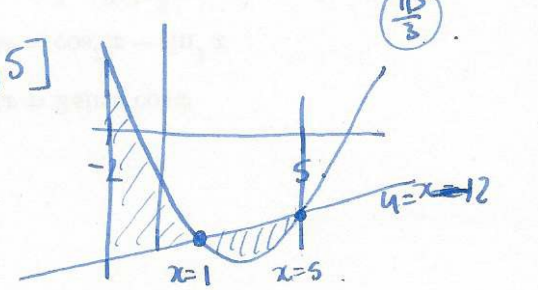


need to solve $f(x) = g(x)$ in $[a, b]$.

Example

- Find area between the graphs $f(x) = x^2 - 5x - 7$ on $[-2, 5]$
  $g(x) = x - 12$

$\frac{113}{3}$.



- Find the area of the region bounded by

$y = \frac{1}{x^2}$, $y = x$, $y = 8x$