Math 232 Calculus 2 Spring 15 Midterm 2b

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes, but no phones or other notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) Find $\int \sqrt{1-9x^2}\,dx$.

$\sin^2 u + \cos^2 u = 1 \leftrightarrow \cos^2 u = 1 - \sin^2 u$

$x = \frac{1}{3}\sin u \qquad \frac{dx}{du} = \frac{1}{3}\cos u$

$\int \sqrt{1-\sin^2 u}\,\frac{dx}{du}\,du = \int \cos u \,\frac{1}{3}\cos u\,du = \frac{1}{3}\int \cos^2 u\,du$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= \frac{1}{3}\int \frac{1}{2}\cos 2u + \frac{1}{2}\,du = \frac{1}{12}\sin 2u + \frac{1}{2}u + c$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{1}{6}\sin u\cos u + \frac{1}{2}\sin^{-1}(3x) + c$

$= \frac{1}{6}3x\sqrt{1-9x^2} + \frac{1}{2}\sin(3x) + c$

$= \frac{1}{2}x\sqrt{1-9x^2} + \frac{1}{2}\sin(3x) + c$

(2) Find $\int \frac{x}{(x+1)^2}\,dx$.

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} = \frac{A(x+1)+B}{(x+1)^2}$$

$x=-1$: $-1 = B$

$x=0$: $0 = A+B \Rightarrow A=1$

$$\int \frac{1}{x+1} - \frac{1}{(x+1)^2}\,dx = \ln|x+1| + \frac{1}{x+1} + C$$

(3) Find $\int_1^2 \frac{1}{x \ln x}\, dx$. $= \lim_{R \to 1} \int_R^2 \frac{1}{x \ln(x)}\, dx$

try: $u = \ln(x)$ $\frac{du}{dx} = \frac{1}{x}$

$\lim_{R \to 0} \int_R^{\ln(2)} \frac{1}{xu} \cdot \frac{dx}{du}\, du = \lim_{R \to 0} \int_R^{\ln(2)} \frac{1}{u}\, du$

$= \lim_{R \to 0} \Big[\ln(u) \Big]_R^{\ln(2)} = \lim_{R \to 0} \ln(\ln(2)) - \ln(R)$

diverges.

$u = x \qquad u' = 1$

$v' = e^{2x} \qquad v = -\frac{1}{2}e^{-2x}$

(4) Find $\int_0^\infty xe^{-2x}\,dx$.

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\lim_{R\to\infty}\int_0^R xe^{-2x}\,dx = \lim_{R\to\infty}\left[-\frac{1}{2}xe^{-2x}\right]_0^R + \frac{1}{2}\int_0^R e^{-2x}\,dx$$

$$= \lim_{R\to\infty}\underbrace{-\frac{1}{2}Re^{-2R}}_{\to 0} - \frac{1}{4}\left[e^{-2x}\right]_0^R = \lim_{R\to\infty}\underbrace{-\frac{1}{4}e^{-2R}}_{\to 0} + \frac{1}{4} = \frac{1}{4}$$

(5) Find the degree three Taylor polynomial centered at $x = 0$ for the function $f(x) = \cos(x^2)$.

$f(x) = \cos(x^2)$ $\qquad f(0) = 1$

$f'(x) = -\sin(x^2).2x$ $\qquad f'(0) = 0$

$f''(x) = -\cos(x^2).4x^2 - \sin(x^2).2$ $\qquad f''(0) = 0$

$f^{(3)}(x) = \sin(x^2).8x^3 - \cos(x^2).8x - \cos(x^2).4x.$ $\qquad f^{(3)}(0) = 0$

$T_3(x) = 1$

(6) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{3n+1}$ converge or diverge?

Diverges as $\lim_{n\to\infty} \frac{(-1)^n n}{3n+1} \neq 0$

$\left(\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0 \right)$

(7) Does the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge? Explain your answer.

diverges.

integral test

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\,dx = \lim_{R\to\infty} \left[\ln(x)\right]_1^R$$

$$= \lim_{R\to\infty} \ln(R) = \infty.$$

(8) (a) Use partial fractions to find an explicit formula for the partial sum
$$s_N = \sum_{n=1}^{N} \frac{1}{4n^2 - 1}.$$

(b) Use your answer to (a) to show that the sum converges, by finding $\lim_{N\to\infty} s_N$.

(9) (a) Find a formula for the partial sum $s_n = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \cdots + \frac{1}{3^n}$, for example by comparing s_n with $\frac{1}{3}s_n$.

(b) Use your answer to (a) to show that the sum converges, by finding $\lim_{N\to\infty} s_N$.

a) $s_n = \frac{1}{3} + \frac{1}{9} + \cdots \frac{1}{3^n}$

$\frac{1}{3} s_n = \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n+1}}$

$s_n - \frac{1}{3}s_n = \frac{1}{3} - \frac{1}{3^{n+1}}$ $\quad$ $\frac{2}{3} s_n = \frac{1}{3} - \frac{1}{3^{n+1}}$ $\quad$ $s_n = \frac{1}{2}\left(1 - \frac{1}{3^n}\right).$

b) $\lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{3^n}\right) = \frac{1}{2}.$

(10) Show the series $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$ converges, by any method.

$$\sum_{n=1}^{\infty} \frac{1}{3^n} + \frac{2^n}{3^n} = \frac{1/3}{1-1/3} + \frac{2/3}{1-1/3} = \frac{1}{2} + 1 = \frac{3}{2}$$

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}$$