recall Laplace transform $f(t) \mapsto L(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Examples

$f(t)$	t^n	e^{at}	$t^n e^{at}$	$\sin(at)$	$\cos(at)$	$t\sin(at)$	$t\cos(at)$
$F(s)$	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-a}$	$\frac{n!}{(s-a)^{n+1}}$	$\frac{a}{s^2+a^2}$	$\frac{s}{s^2+a^2}$	$\frac{2as}{(s^2+a^2)^2}$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

recall $L(cf) = cF(s) \quad L(f+g) = F(s)+G(s)$.

Example $L(e^{at}+e^{bt}) = \frac{1}{s-a} + \frac{1}{s-b} = \frac{2s-a-b}{(s-a)(s-b)} = \frac{2s-a-b}{s^2-(a+b)s+ab}$.

Derivatives
$$L(f') = sF(s) - f(0)$$
$$L(f'') = s^2F(s) - sf(0) - f'(0)$$
$$L(f^{(n)}) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f''(0) - \cdots - f^{(n-1)}(0).$$

Example solve IVP $y''+4y'+3y = e^t \quad y(0)=0,\ y'(0)=2$

$$L(y'') + 4L(y') + 3L(y) = L(e^t)$$

$$s^2Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{2} + 4\big(sY - \underbrace{y(0)}_{0}\big) + 3Y = \frac{1}{s-1}$$

$$Y\underbrace{(s^2+4s+3)}_{(s+3)(s+1)} - 2 = \frac{1}{s-1} \qquad Y(s+3)(s+1) = \frac{1}{s-1} + 2 = \frac{2s-1}{s-1}.$$

$$Y = \frac{2s-1}{(s-1)(s+1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3}$$

partial fractions: $2s-1 = A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1)$

$s=1$: $1 = 8A \quad A = 1/8$

$s=-1$: $-3 = -4B \quad B = 3/4$

$s=-3$: $-7 = 8C \quad C = -7/8$

$$Y(s) = \frac{1/8}{s-1} + \frac{-3/4}{s+1} + \frac{-7/8}{s+3}.$$

$$y(t) = \frac{1}{8}e^t - \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}$$

§3.3 Shifts and the Heaviside function

Thm $L(e^{at}f(t)) = F(s-a)$

Proof $L(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$ $\square$

Example $L(\cos(bt)) = \dfrac{s}{s^2+b^2}$ $\quad L(e^{at}\cos(bt)) = \dfrac{s-a}{(s-a)^2+b^2}$

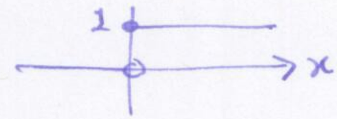
inverse Laplace transforms $L^{-1}(F(s-a)) = e^{at}f(t)$

Example $L^{-1}\left(\dfrac{4}{s^2+4s+20}\right) = L^{-1}\left(\dfrac{4}{(s+2)^2+16}\right) = F(s+2)$

where $F(s) = \dfrac{4}{s^2+16} \Rightarrow f(t) = \sin(4t)$ so $L^{-1}\left(\dfrac{4}{s^2+4s+20}\right) = e^{-2t}\sin(4t)$.

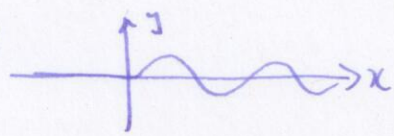
Heaviside/step function

Def^n The Heaviside function is $H(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$



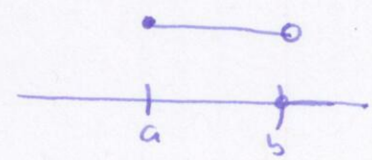
shifted Heaviside: $H(t-a) = \begin{cases} 0 & t<a \\ 1 & t\geq a \end{cases}$

Examples $H(t)\sin(t)$



Pulse functions $H(t-a) - H(t-b) = \begin{cases} 0 & t<a \\ 1 & a\leq t < b \\ 0 & t\geq b \end{cases}$

$a<b$.



Q: what is $L(H)$?

$L(H)(s) = \int_0^\infty e^{-st}H(t)\,dt = \int_0^\infty e^{-st}\,1\,dt = \dfrac{1}{s}$

$L(H(t-a)) = \int_0^\infty e^{-st}H(t-a)\,dt = \int_a^\infty e^{-st}\,dt = \left[-\dfrac{1}{s}e^{-st}\right]_a^\infty = \dfrac{1}{s}e^{-as}$

more generally: Thm

$$L(H(t-a)f(t-a)) = \int_0^\infty e^{-st} H(t-a) f(t-a)\,dt$$ Proof: sub $u=t-a$ $\frac{du}{dt}=1$

$$\int_a^\infty e^{-s(u+a)} f(u)\,du = e^{-as}\int_a^\infty e^{-su} f(u)\,du = e^{-as} F(s). \quad \Box.$$

Reverse: $L^{-1}(e^{-as}F(s)) = H(t-a)f(t-a)$

Examples ① find $L(f)$ where $f(t) = \begin{cases} 0 & t \le 2 \\ t^2+1 & t \geq 2 \end{cases}$ $f(t) = H(t-2)(t^2+1)$

write in terms of $t-2$

$$f(t) = H(t-2)\left(((t-2)+2)^2+1\right) = H(t-2)\left((t-2)^2+4(t-2)+5\right) = H(t-2)g(t-2)$$

$g(t) = t^2+4t+5.$

$$\text{so } L(f) = e^{-2s} L(t^2+4t+5) = e^{-2s}\left(L(t^2)+4L(t)+5L(1)\right)$$

$$= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{5}{s}\right)$$

② find $L^{-1}\left(\frac{se^{-3s}}{s^2+4}\right)$ use: $L^{-1}\left(\frac{s}{s^2+4}\right) = \cos(2t)$

so $= H(t-3)\cos(2(t-3)).$

③ solve $y''+4y=f(t)$ $y(0)=0, y'(0)=0$ $f(t) = \begin{cases} 0 & t<3 \\ t & t\geq 3 \end{cases}.$

$= H(t-3)t$

$$L(y'')+4L(y) = L(f)$$

$$s^2Y - \underbrace{sy(0)}_{0} - \underbrace{y'(0)}_{0} + 4Y = L(f) = L(H(t-3)(t-3+3))$$

$$= L(H(t-3)(t-3)) + 3L(H(t-3)).$$

$$\underbrace{s^2Y+4Y}_{(s^2+4)Y} = \frac{e^{-3s}}{s^2} + \frac{3e^{-3s}}{s} = e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right) = e^{-3s}\left(\frac{1+3s}{s^2}\right).$$

$$Y = e^{-3s}\frac{(1+3s)}{s^2(s^2+4)}$$

partial fractions: $Y = e^{-3s}\dfrac{1+3s}{s^2(s^2+4)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+4}$

$$Y = e^{-3s}\left[\frac{3/4}{s} + \frac{1/4}{s^2} + \frac{-3/4 s - 1/4}{s^2+4}\right] \qquad \frac{-3/4 s}{s^2+4} + \frac{-1/4}{s^2+4}$$

$$y(t) = \frac{3}{4}H(t-3) + \frac{1}{4}H(t-3)(t-3) - \frac{3}{4}H(t-3)\cos(2(t-3)) - \frac{1}{8}H(t-3)\sin(2t-3)$$

$$= H(t-3)\left[\frac{3}{4} + \frac{1}{4}(t-3) - \frac{3}{4}\cos(2(t-3)) - \frac{1}{8}\sin(2(t-3))\right].$$

Heaviside's formula $F(s) = \dfrac{p(s)}{q(s)}$ $\quad \deg(q) > \deg(p)$, no common roots

$$q(s) = c(s-a_1)(s-a_2)\cdots(s-a_n)$$

let $q_i(s) = \dfrac{q(s)}{(s-a_i)}$. Then $\mathcal{L}^{-1}(F) = \displaystyle\sum_{j=1}^{n} \frac{p(a_j)}{q_j(a_j)} e^{a_j t}$

Example $F(s) = \dfrac{s}{(s^2+4)(s-1)} = \dfrac{s}{(s-2i)(s+2i)(s-1)}$ $\quad a_1 = 2i \quad a_2 = -2i \quad a_3 = 1$

$$\mathcal{L}^{-1}(F) = \frac{2i}{4i(2i-1)}e^{2it} + \frac{-2i}{-4i(-2i-1)}e^{-2it} + \frac{1}{(1-2i)(1+2i)}e^{t} = -\frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) + \frac{1}{5}e^{t}$$

§3.4 Convolution

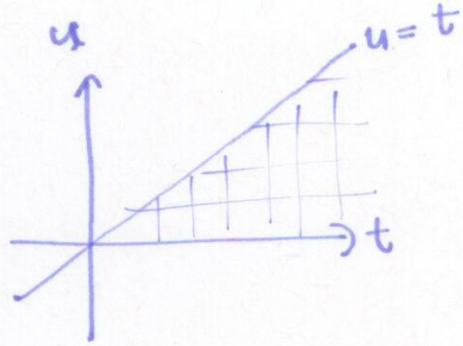
Def^{n} $f(t), g(t)$ defined for $t \geq 0$, then $f*g$ is

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau \quad \text{for } t \geq 0 \text{ s.t. this integral converges.}$$

Th^{m} $L(f*g) = L(f).L(g)$ equivalently, $L(f*g) = F(s)G(s)$

inverse version: $L^{-1}(FG) = f*g$

Proof $(f*g)(t) = \int_0^t f(t-u)\,g(u)\,du$



$$L(f*g) = \int_0^\infty e^{-st} \int_0^t f(t-u)\,g(u)\,du\,dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(t-u)\,g(u)\,du\,dt$$

change order of integration

$$= \int_0^\infty \int_u^\infty e^{-st} f(t-u)\,g(u)\,dt\,du$$

$$= \int_0^\infty g(u) \int_u^\infty e^{-st} f(t-u)\,dt\,du \qquad v = t-u \quad \frac{dv}{dt} = 1.$$

$$= \int_0^\infty g(u) \int_0^\infty e^{-s(u+v)} f(v)\,dv\,du.$$

$$= \int_0^\infty g(u)\,e^{-su} \underbrace{\int_0^\infty e^{-sv} f(v)\,dv}_{F(s)}\,du$$

$$= F(s) \int_0^\infty g(u)\,e^{-su}\,du = F(s)G(s). \quad \square.$$

Example find $L^{-1}\left(\frac{1}{s(s-1)}\right)$ $\quad F(s) = \frac{1}{s} \quad G(s) = \frac{1}{s-1}. \quad f(t) = 1 \quad g(t) = e^t$

$$L^{-1}(FG) = f(t)*g(t)$$

$$= \int_0^t 1\,\cancel{g(t-u)}\,g(u)\,du = \int_0^t e^u\,du$$

$$= \left[e^u\right]_0^t = e^t - 1.$$