$f(t) = e^{at}$ $\quad F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^\infty$

$= \lim_{t\to\infty} \frac{1}{a-s} e^{(a-s)t} - \frac{1}{a-s} = \frac{1}{s-a} \quad (s>a).$

Fact $f(t) = t^n e^{at}$ $\quad F(s) = \frac{n!}{(s-a)^{n+1}}$

$f(t) = \sin(at)$ $\quad F(s) = \int_0^\infty e^{-st} \sin at\,dt$

$F(s) = \frac{a}{s^2+a^2}$

$= \left[-\frac{1}{s} e^{-st} \sin at\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} a\cos at\,dt.$

$= \frac{1}{s}\left[-\frac{1}{s} e^{-st} \cos at\right]_0^\infty - \frac{1}{s^2}\int_0^\infty e^{-st} a^2 \sin at\,dt.$

$F(s) = \frac{a}{s^2} - \frac{a^2}{s^2}F(s).$

$F(s)\left(1+\frac{a^2}{s^2}\right) = \frac{a}{s^2}$ $\quad F(s)\left[\frac{s^2+a^2}{s^2}\right] = \frac{a}{s^2}.$

$F(s) = \frac{a}{s^2+a^2}.$

Fact $f(t) = \cos(at)$ $\quad F(s) = \frac{s}{s^2+a^2}$ $\quad e^{at}\sin bt \quad \frac{b}{(s-a)^2+b^2}$

$t \sin at \quad \frac{2as}{(s^2+a^2)^2}$ $\quad e^{at}\cos bt \quad \frac{s-a}{(s-a)^2+b^2}$

$t\cos at \quad \frac{s^2-a^2}{(s^2+a^2)^2}$

L : function $\longrightarrow$ function.   claim L is linear.

$L(cf) = cL(f)$ $\int_0^\infty e^{-st} cf(t)\,dt = c\int_0^\infty e^{-st} f(t)\,dt.$

$L(f+g) = L(f) + L(g)$ $\int_0^\infty e^{-st}(f(t)+g(t))\,dt = \int_0^\infty e^{-st} f(t)\,dt + \int_0^\infty e^{-st} g(t)\,dt.$

now do $L(e^{at}+e^{bt}) = \frac{a-b}{(s-a)(s-b)}$

there is an inverse Laplace transform L^{-1} : {functions} $\to$ {functions}

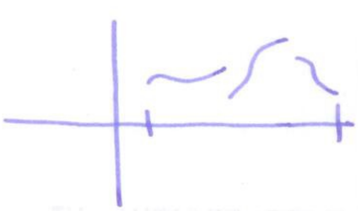
§3.2 Solving initial value problems

Def^n $f(t)$ defined on $[a,b]$ is piecewise continuous on $[a,b]$ if

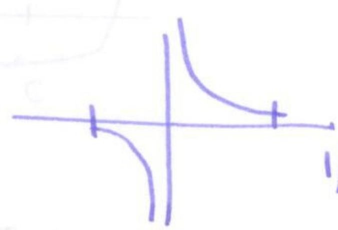
1. f is continuous at all but finitely many points of $[a,b]$
2. if f is not continuous at $t_0 \in (a,b)$ then $f(t)$ has finite limits from both sides of t_0
3. $f(t)$ has finite limits as $t \to a$ and $t - b$.

Note f has finitely many jump discontinuities, where the size of the jump is $\left| \lim_{t\to t_0+} f(t) - \lim_{t\to t_0-} f(t) \right|$

Example



Non example



$1/x$ on $[-1,1]$.

Thm Laplace transform for derivatives

f continuous for $t \geq 0$

f' piecewise continuous on $[0,k]$ for every $k > 0$

$\lim_{k\to\infty} e^{-sk} f(k) = 0$ if $s > 0$

Then: $L(f')(s) = sF(s) - f(0)$

if jump discontinuity at 0: $L(f')(s) = sF(s) - f(0+)$.

$f(0+) = \lim_{x\to 0+} f(x)$

Proof

integration by parts

$$L(f')(s) = \int_0^\infty e^{-st} f'(t)\,dt = \Big[e^{-st} f(t)\Big]_0^\infty - \int_0^\infty -s\, e^{-st} f(t)\,dt$$

$$= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\, \underbrace{L(f)(s)}_{F(s)}$$

$$= sF(s) - f(0) \quad \square.$$

Thm Laplace transforms for higher derivatives.

$f, f', \ldots, f^{(n-1)}$ continuous for $t \geq 0$

$f^{(n)}$ piecewise cts on $[0,k]$ for all k.

$\lim_{k\to\infty} e^{-sk} f^{(i)}(k) = 0$ for $s>0$ and $1 \leq i \leq n-1$

then $L(f^{(n)})(s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

Special case $n=2$

$$L(f'')(s) = s^2 F(s) - s f(0) - f'(0).$$

Example solve $y' - 4y = 1$, $y(0) = 1$

Laplace transform: $L(y' - 4y) = L(1) = \frac{1}{s}$

$$L(y') - 4L(y)$$

$$sL(y) - y(0)$$

so: $sY(s) - y(0) - 4Y(s) = \frac{1}{s}$

$$(s-4)Y(s) = \frac{1}{s} + y(0) = 1 + \frac{1}{s}$$

so $Y(s) = \frac{1}{s-4} + \frac{1}{s(s-4)}$

now $y(t) = L^{-1}(Y(s))$.

$$= \underbrace{L^{-1}\left(\frac{1}{s-4}\right)}_{=e^{4t}} + \underbrace{L^{-1}\left(\frac{1}{s(s-4)}\right)}_{-\frac{1}{4}(e^{0t} - e^{4t})}$$

$$y(t) = e^{4t} + \frac{1}{4}(e^{4t} - 1) = \frac{5}{4}e^{4t} - \frac{1}{4}.$$