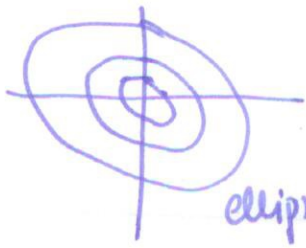
complex eigenvalues

pure imaginary $\lambda = \pm \beta i$ general solution $\underline{X} = c_1(\underline{a}\cos(\beta t) - \underline{b}\sin\beta t) + c_2(\underline{a}\sin\beta t + \underline{b}\cos\beta t)$.

(ellipses).

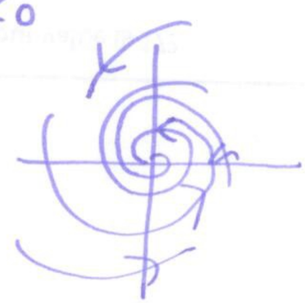


ellipses.

non-zero real part $\lambda = \alpha \pm \beta i$ general solution $\underline{X} = e^{\alpha t}(\longrightarrow)$.

$\alpha > 0$ $\alpha < 0$

spiral out



spiral in.

Example (predator/prey model)

two species with population $x(t)$ rabbits
$y(t)$ foxes

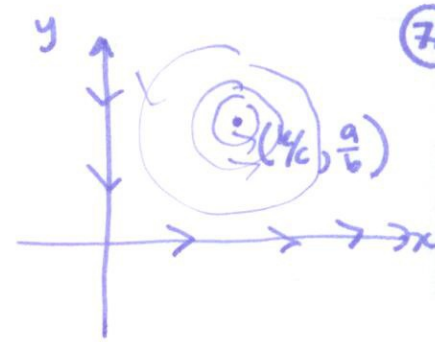
rate of change of rabbit population = baby rabbits − eaten rabbits

$x'(t)$ $ax(t)$ $-bx(t)y(t)$

rate of change of fox population = eaten rabbits − starving foxes/death rate

$y'(t)$ $cx(t)y(t)$ ky

$$\left.\begin{aligned} x' &= ax - bxy \\ y' &= cxy - ky \end{aligned}\right\} \text{ 2x2 system but \underline{non-linear}.}$$



special cases

$x=0$ (no rabbits) $y' = -ky$ $y = ce^{-kt}$

$y=0$ (no foxes) $x' = ax$ $y = ce^{at}$

① find critical points, i.e. solve $\begin{bmatrix} x' \\ y' \end{bmatrix} = X' = \underline{0}$.

$$\left.\begin{aligned} ax - bxy &= 0 \\ cxy - ky &= 0 \end{aligned}\right\} \;(*)\quad \begin{aligned} x(a-by) &= 0 \\ y(cx-k) &= 0 \end{aligned} \quad \left.\begin{aligned} y &= a/b \\ x &= k/c \end{aligned}\right\}$$

equilibrium solution, i.e. $x = k/c$, $y = a/b$ is a solution to $(*)$

② linearize near critical point

recall (1 var) $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \cdots$

$F(X) \approx F(X_0) + DF(X_0)(X - X_0) + \cdots$ recall $F(X) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$ $DF(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$.

$F(X) = \begin{bmatrix} ax - bxy \\ cxy - ky \end{bmatrix}$ $DF(X) = \begin{bmatrix} a - by & -bx \\ cy & cx - k \end{bmatrix}$ $DF(X_0) = \begin{bmatrix} 0 & -\frac{bk}{c} \\ \frac{ca}{b} & 0 \end{bmatrix}$

$X' \approx DF(X_0)(X - X_0)$

$(X - X_0)' \approx DF(X_0)(X - X_0)$

$Y' = DF(X_0)Y$ eigenvalues: $\det(A - \lambda I) = 0$ $\begin{vmatrix} -\lambda & -\frac{bk}{c} \\ \frac{ac}{b} & -\lambda \end{vmatrix} = \lambda^2 + ck = 0$

$\lambda = \pm\sqrt{ck}\, i$ periodic critical point.

Example (competing species)

$x(t)$ sheep rate of change of sheep = baby sheep − resource completion.

$y(t)$ cows

$$x'(t) = ax - bxy.$$
$$y'(t) = ky - cxy.$$

$X' = F(X)$ $\quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax - bxy \\ hy - cxy \end{bmatrix}$

① find critical points, solve $X' = F(X) = \underline{0}$

$x(a - by) = 0 \qquad y = a/b$

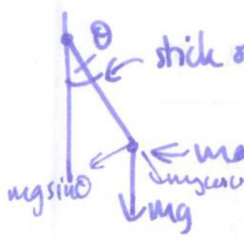
$y(h - cx) = 0 \qquad x = h/c$

② linearize near critical point

$F(X) \simeq F(X_0) + DF(X_0)(X - X_0)$ $\qquad DF(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} a - by & -bx \\ -cy & h - cx \end{bmatrix}.$

$DF(X_0) = \begin{bmatrix} 0 & -\frac{bh}{c} \\ -\frac{ac}{b} & 0 \end{bmatrix}$ $\qquad$ eigenvalues $\lambda^2 - hc = 0 \quad \lambda = \pm\sqrt{hc}$, saddle.

<u>Example</u> (pendulum)



stick of length l $\qquad \ddot{\theta} = -\frac{g}{l}\sin\theta$

<u>note</u> θ small $\sin\theta \simeq \theta$ so get simple harmonic motion.

① reduce to first order: $\left.\begin{array}{l} \theta = x \\ \dot{\theta} = y \end{array}\right\}$ $\quad x' = y$

$y' = -\frac{g}{l}\sin x = -\omega^2 \sin x$

$X' = F(X) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -\frac{g}{l}\sin x \end{bmatrix} = \begin{bmatrix} y \\ -\omega^2 \sin x \end{bmatrix}.$

② find critical points: $X' = F(X) = 0 \qquad y = 0$

$-\omega^2 \sin x = 0 \Rightarrow x = 2\pi n \qquad (2\pi n, 0).$

③ linearize at critical points

$DF(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos x & 0 \end{bmatrix}$