$(A-4I)\underline{v_2} = \underline{v_1}$

$$\begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \underline{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left[\begin{array}{cc|c} -3 & 3 & 1 \\ -3 & 3 & 1 \end{array}\right] = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} -3 & 3 & 1 \\ 0 & \underbrace{0}_{s} & 0 \end{array}\right] \end{array} \qquad \cancel{\underline{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} s}$$

homogeneous solution : $\begin{bmatrix} 1 \\ 1 \end{bmatrix} s$

$x_2 = s$

$-3x_1 + 3x_2 = 1$

$x_1 = s - 1/3.$

$\underline{v_2} = \begin{bmatrix} \\ 1 \end{bmatrix} s$

particular solution : $\begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$

so $\underline{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$ just need one, can choose $s=0$, $\underline{v_2} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$.

so 2nd solution is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} e^{4t}$

general soln : $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{4t} + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} e^{4t} \right).$

<u>Example</u> $X' = AX$ $\quad A = \begin{bmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ eigenvalues : $-2, -2, -2$

eigenvector : $\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}$.

so one solution is $\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} e^{-2t}$ know the solutions contain te^{-2t}, $t^2 e^{-2t}$.

<u>method</u> :

look for second solution of the form $\underline{X_2} = \underline{v_1} te^{-2t} + \underline{v_2} e^{-2t}$

third $\underline{X_3} = \frac{1}{2}\underline{v_1} t^2 e^{-2t} + \underline{v_2} e^{-2t} + \underline{v_3} e^{-2t}$.

plug in X_2: $\quad X_2' = AX_2$

$$\underline{v_1}\left(e^{-2t} - \cancel{2te^{-2t}}\right) + \underline{v_2}\left(-2e^{-2t}\right) = \underbrace{\cancel{A\underline{v_1}te^{-2t}}}_{-2\underline{v_1}} + A\underline{v_2}e^{-2t}$$

$\underline{v_1} - 2\underline{v_2} = A\underline{v_2}$ $\qquad (A+2I)\underline{v_2} = \underline{v_1}$

$$\left[\begin{array}{ccc|c} 0 & -1 & -5 & -1 \\ 25 & -5 & 0 & -5 \\ 0 & 1 & 5 & 1 \end{array}\right]$$ homogeneous solution $\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}$

particular solution $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$$\rightsquigarrow \left[\begin{array}{ccc|c} 5 & -1 & 0 & -1 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ general solution $\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can choose $s=0$.

↑ choose free variable $s=0$

$x_2 + 0 = 1 \Rightarrow x_2 = 1.$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$5x_1 - \underbrace{x_2}_{1} + \underbrace{0x_3}_{0} = -1 \Rightarrow x_1 = 0.$

so 2nd solution is $\overset{\underline{v}_1}{\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}} te^{-2t} + \overset{\underline{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} e^{-2t}$

now look for $X_3 = \frac{1}{2}\underline{v}_1 t^2 e^{-2t} + \underline{v}_2 te^{-2t} + v_3 e^{-2t}$

plug in to: $X' = AX$

$$\frac{1}{2}\underline{v}_1\left(\cancel{2te^{-2t}} + t^2 \cdot -2e^{-2t}\right) + \underline{v}_2\left(e^{-2t} + \cancel{t \cdot -2e^{-2t}}\right) + \underline{v}_3\left(-2e^{-2t}\right)$$

$$= \underbrace{\frac{1}{2}A\underline{v}_1}_{-2\underline{v}_1} t^2 e^{-2t} + \cancel{\underbrace{A\underline{v}_2}_{\underline{v}_1 - 2\underline{v}_2} te^{-2t}} + A\underline{v}_3 e^{-2t}.$$

leaves: $e^{-2t}\left[\ \underline{v}_2 - 2\underline{v}_3 = A\underline{v}_3\ \right]$

$(A + 2I)\underline{v}_3 = \underline{v}_2$

$$\left[\begin{array}{ccc|c} 0 & -1 & -5 & 0 \\ 25 & -5 & 0 & 1 \\ 0 & 1 & 5 & 0 \end{array}\right]$$ homogeneous solution: $\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$

particular solution $\cancel{\begin{bmatrix} 0 \\ 1/25 \\ 0 \end{bmatrix}}$. $\begin{bmatrix} 1/25 \\ 0 \\ 0 \end{bmatrix}$. so ↑

$$\boxed{X_3 = \frac{1}{2}\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} t^2 e^{-2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} 1/25 \\ 0 \\ 0 \end{bmatrix} e^{-2t}}$$

§10.3 Non-homogeneous case $X' = AX + G$

§10.3.1 Variation of parameters.

$$X' = AX + G$$

solve $X' = AX$, let solution be $\Omega(t).\underline{C}$.

fundamental matrix of solutions.

Look for a solution $\Psi(t) = \Omega(t)\,U(t)$

$\uparrow$ $n\times 1$ vector.

$$\Psi' = \Omega' U + \Omega U'$$

plug in:

$$\underbrace{\underbrace{\Omega' U}_{} + \Omega U' = \underbrace{A\Omega U}_{}}_{\text{cancel}} + G$$

$$\Omega U' = G$$

Ω invertible as columns linearly independent!

$$U' = \Omega^{-1} G$$

$$U(t) = \int \Omega^{-1}(t) G(t)\, dt$$

general solution: ~~U(t)~~ $X(t) = \Omega(t).C + \Omega(t)\,U(t)$

<u>Example</u> $X' = \begin{bmatrix} 1 & -10 \\ -1 & 4 \end{bmatrix} X + \begin{bmatrix} t \\ 1 \end{bmatrix}$ eigenvalues -1 $\quad 6$

eigenvectors $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

fundamental matrix $\Omega(t) = \begin{bmatrix} 5e^{-t} & -2e^{+6t} \\ e^{-t} & e^{+6t} \end{bmatrix}$

$U(t) = \int \Omega^{-1}(t) G(t) dt$ $\qquad \Omega^{-1}(t) = \dfrac{1}{5e^{5t} + 2e^{5t}} \begin{bmatrix} e^{6t} & 2e^{6t} \\ -e^{-t} & 5e^{-t} \end{bmatrix}$

$$= \frac{1}{7} \begin{bmatrix} e^{t} & 2e^{t} \\ -e^{-6t} & 5e^{-6t} \end{bmatrix}$$

$$u(t) = \int \Omega^{-1}(t)G(t)\,dt = \int \frac{1}{7}\begin{bmatrix} e^{t} & 2e^{t} \\ -e^{-6t} & 5e^{-6t} \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix} dt$$

$$= \frac{1}{7}\int \begin{bmatrix} te^{t} + 2e^{t} \\ -te^{-6t} + 5e^{-6t} \end{bmatrix} dt = \frac{1}{7}\begin{bmatrix} te^{t} - \int e^{t}\,dt + 2e^{t} \\ -t\cdot -\frac{1}{6}e^{-6t} - \frac{1}{6}\int e^{-6t}\,dt - \frac{5}{6}e^{-6t} \end{bmatrix}.$$

$$= \frac{1}{7}\begin{bmatrix} e^{t}(t+1) \\ e^{-6t}\left(\frac{t}{6} + \frac{1}{36} - \frac{5}{6}\right) \end{bmatrix} = \frac{1}{7}\begin{bmatrix} e^{t}(t+1) \\ e^{-6t}\left(-\frac{29}{36}\right) \end{bmatrix}$$

general solution: $\Omega(t).C + \cancel{\Omega(t)}X_p(t)$

$$= \begin{bmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \cancel{\frac{1}{7}\begin{bmatrix} e^{t}(t+1) \\ e^{-6t}\left(-\frac{29}{36}\right) \end{bmatrix}}\begin{bmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{bmatrix}\frac{1}{7}\begin{bmatrix} e^{t}(t+1) \\ e^{-6t}\left(-\frac{29}{36}\right) \end{bmatrix}.$$

$$'' \qquad + \frac{1}{3}\begin{bmatrix} 17/6 + (49/7)t \\ 1/12 \quad t/2 \end{bmatrix}.$$

§10.3.2 Diagonalization

want to solve $X' = AX + G$ where $A = PDP^{-1}$

change coordinates: set $X = PZ$

Example $X' = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} X + \begin{bmatrix} 8 \\ 4e^{3t} \end{bmatrix}$ eigenvalues $2 \quad 6$

eigenvector $\begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

so $P = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$ and $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$

set $X = PZ$ plug in: $X' = PZ'$

$$PZ' = APZ + G$$

multiply by P^{-1}: $\quad P^{-1}PZ' = P^{-1}APZ + P^{-1}G \qquad P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & -3 \end{bmatrix} \frac{1}{-4}$

$$Z' = DZ + P^{-1}G$$

i.e. $\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 4e^{3t} \end{bmatrix}$

two independent linear first order equations: $\quad z_1' = 2z_1 - 2 + e^{3t}$

$$z_2' = 6z_2 + 2 + 3e^{3t}$$

solve: $\quad z_1 = c_1 e^{2t} + e^{3t} + 1$

$$z_2 = c_2 e^{6t} - e^{3t} - \frac{1}{3}$$

then $\quad X(t) = PZ(t) = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} c_1 e^{2t} \\ c_2 e^{6t} \end{bmatrix} + \begin{bmatrix} e^{3t} + 1 \\ -e^{3t} - 1/3 \end{bmatrix} \right).$

$$= \begin{bmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -4e^{3t} - 10/3 \\ 2/3 \end{bmatrix}$$

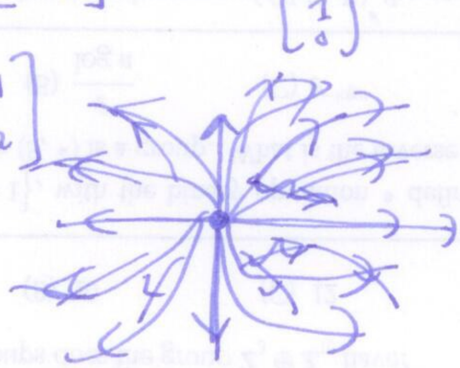
§10.6 Phase portraits.

local pictures for solutions of $X' = AX \qquad A$ constant matrix.

special case $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \lambda_1 = 3 \qquad \lambda_2 = 2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

solutions $\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$



Now spose A has positive real eigenvalues $\lambda_1 > \lambda_2 > 0$

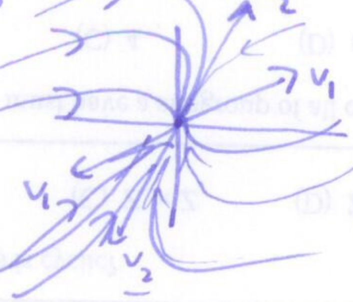
eigenvector $\underline{v}_1$ $\underline{v}_2$

$X' = AX$ has general solution $c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t}$

sink.

negative eigenvalues $\lambda_1 < \lambda_2 < 0$

as above; arrows reversed.

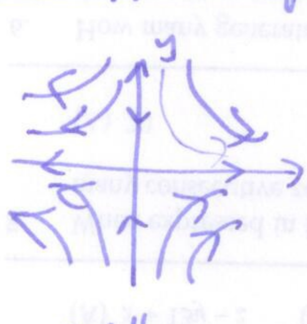


source.

real eigenvalues opposite sign $\lambda_1 < 0 < \lambda_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

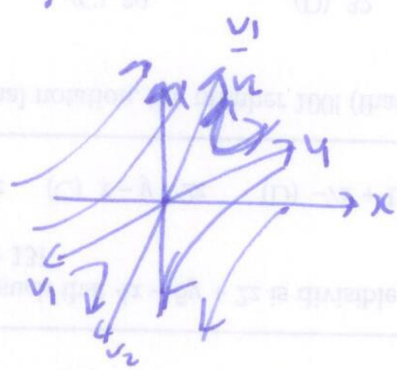
soln $\begin{bmatrix} c_1 e^{t} \\ c_2 e^{-t} \end{bmatrix}$



saddle

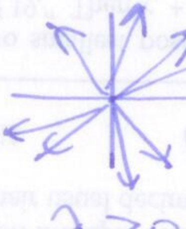
in general $\lambda_1 < 0 < \lambda_2$

$\underline{v}_1$ $\underline{v}_2$



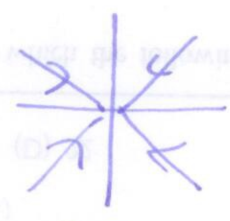
real eigenvalues equal multiplicity 2

(1) two eigenvectors $\lambda > 0$



$\lambda > 0$

or



$\lambda < 0$

(2) only one eigenvector

recall $X' = AX$ has general solution $\underline{X} = c_1 \underline{v}_1 e^{\lambda t} + (c_1 v_2 + c_2 \underline{v}_1) e^{\lambda t}$

$= c_1 (\underline{v}_1 t + v_2) e^{\lambda t} + c_2 \underline{v}_1 e^{\lambda t}$