<u>observation</u> <u>recall</u> $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$

so $e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$ and $\frac{d}{dt}(e^{At}) = Ae^{At}$

so $X' = AX$ has solution $e^{At} \cdot \underline{c}_0$.

---

<u>Thm</u> $X' = AX$ $\underset{n\times n}{A} = [a_{ij}(t)]$ cts on an open interval $I$. Then

1. there are exactly $n$ linearly independent solns on $I$.
2. every solution is a linear combination of these.

---

<u>Q</u>: what happens if $A$ is not diagonalizable?

<u>Example</u>. $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

method ①: $X' = AX$ $\quad x_1' = \lambda x_1 + x_2 \quad \Rightarrow x_1' = \lambda x_1 + c_2 e^{\lambda t}$

$x_2' = \lambda x_2 \quad \Rightarrow x_2 = c_2 e^{\lambda t} \quad \Rightarrow x_1 = c_1 e^{\lambda t} + \cancel{c}te^{\lambda t}$.

<u>try</u> $x_1 = ate^{\lambda t}$

$x_1' = ae^{\lambda t} + a\lambda t e^{\lambda t} = \cancel{\lambda} ate^{\lambda t} + c_2 e^{\lambda t} \Rightarrow a = c_2$.

general solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda t} + c_2 te^{\lambda t} \\ c_2 e^{\lambda t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda t} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix} e^{\lambda t}$

method ②: $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$ $\quad A^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}$ $\quad A^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots = \begin{bmatrix} 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots & t + 2\lambda\frac{t^2}{2!} + 3\lambda^2\frac{t^3}{3!} + \cdots \\ 0 & 1 + \lambda t + \lambda^2\frac{t^2}{2!} + \cdots \end{bmatrix}.$$

$$= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}.$$

<u>Example</u> $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}$ $\quad A^3 = \begin{bmatrix} \lambda^3 & 3\lambda^2 & 3\lambda \\ 0 & \lambda^3 & 3\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}$

$$A^4 = \begin{bmatrix} \lambda^4 & 4\lambda^3 & 6\lambda^2 \\ 0 & \lambda^4 & 4\lambda^3 \\ 0 & 0 & \lambda^4 \end{bmatrix} \qquad A^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{1}{2}n(n-1)\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}.$$ (62)

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \dots = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

## §10.1.2 Non-homogeneous systems

$X' = AX + G$

① solve homogeneous system $X' = AX$. $X_h = e^{At} \cdot \underline{c}$.

② find particular solution $\underline{X}_p$

general solution is $\underline{X} = \underline{X}_h + \underline{X}_p$.

## §10.2 Homogeneous case

want to solve $X' = AX$ where $A = TJT^{-1}$

solution is $X = e^{At} \cdot \underline{c}$ or $Te^{Jt} \cdot \underline{c}$

Q: what is $T$?

A: change of basis matrix, i.e. matrix of eigenvalues.

so solution is $c_1 \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} + \dots + c_n \underline{v_n} e^{\lambda_n t}$.

where $\underline{v_1}, \dots, \underline{v_n}$ is a basis of eigenvectors with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$.

## § 10.2.1 Complex eigenvalues

**Example** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues: $\lambda_1 = i \quad \underline{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_2 = -i \quad \underline{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

**soln** $c_1 \underline{v}_1 e^{it} + c_2 \underline{v}_2 e^{-it}$ (complex! want real solutions).

**recall** . $e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$.

. roots of real polynomials come in complex conjugate pairs $\alpha \pm i\beta$.

complex solution $c_1 \underbrace{\underline{v}}_{\underline{a}+i\underline{b}} e^{(\alpha+i\beta)t} + c_2 \underbrace{\overline{\underline{v}}}_{\underline{a}-i\underline{b}} e^{(\alpha-i\beta)t}$

gives real solutions:

$$\underline{a}\, e^{\alpha t} \underbrace{\left(e^{i\beta t} + e^{-i\beta t}\right)}_{2\cos\beta t} + i\underline{b}\, e^{\alpha t} \underbrace{\left(e^{it} - e^{-it}\right)}_{2i\sin\beta t}$$

so two independent solutions are: $e^{\alpha t}\left[\underline{a}\cos\beta t - \underline{b}\sin\beta t\right]$ $c_1 = c_2$.

and $e^{\alpha t}\left[\underline{a}\sin\beta t + \underline{b}\cos\beta t\right]$. $c_1 = -c_2$.

**Alternatively** consider $e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots$

$= T\left(I + Dt + D^2\frac{t^2}{2!} + \cdots\right)T^{-1} \qquad D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

$e^{it} = 1 + it + \frac{(it)^2}{2!} + \cdots = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots = \cos t - i\sin t$ etc...

$= i\left(t - \frac{t^3}{3!} + \cdots\right)$

Example $X' = AX$ $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ eigenvalues $2, -1 \pm \sqrt{3}i$

eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2\sqrt{3}i \\ -3+\sqrt{3}i \end{bmatrix} \begin{bmatrix} 1 \\ 2\sqrt{3}i \\ -3-\sqrt{3}i \end{bmatrix}$

solutions: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}$ $\begin{bmatrix} 1 \\ -2\sqrt{3}i \\ -3+\sqrt{3}i \end{bmatrix} e^{(-1+\sqrt{3})t}$ $\begin{bmatrix} 1 \\ 2\sqrt{3}i \\ -3-\sqrt{3}i \end{bmatrix} e^{(-1-\sqrt{3})t}$.

$$= \underbrace{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}}_{\underline{a}} + i \underbrace{\begin{bmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{bmatrix}}_{\underline{b}}$$

real solns:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}, \quad e^{-t}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \cos(\sqrt{3}t) - \begin{bmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{bmatrix} \sin(\sqrt{3}t)\right), \quad e^{-t}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \sin(\sqrt{3}t) + \begin{bmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{bmatrix} \cos(\sqrt{3}t)\right)$$

## §10.2.2 Not enough eigenvalues

Example $X' = AX$ $A = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix}$ eigenvalues 4, 4 eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{v}_1$

so one solution is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ ~~Other~~ look for solution is ~~$\underline{v}_2 te^{4t}$~~ $\underline{v}_1 te^{4t} + \underline{v}_2 e^{4t}$

plug in: $X' = AX$

$$\cancel{\underline{v}_2 \left(e^{4t} + 4te^{4t}\right) = A\underline{v}_2 te^{4t}}$$

$$\underline{v}_1 \left(e^{4t} + 4te^{4t}\right) + \underline{v}_2 4e^{4t} = \underbrace{A\underline{v}_1}_{4\underline{v}_1} te^{4t} + A\underline{v}_2 e^{4t}$$

cancel.

$$e^{4t}\left[\underline{v}_1 + 4\underline{v}_2\right] = e^{4t}\left[A\underline{v}_2\right]$$

$$A\underline{v}_2 - 4\underline{v}_2 = \underline{v}_1$$