§ 9.3 Special matrices

Defn An $n\times n$ matrix A is orthogonal if $A^{-1} = A^T$

$(\Rightarrow \quad AA^T = A^TA = I)$

Thm If A is orthogonal then $|A| = \pm 1$

Proof $|A| = |A^T|$ so $|AA^T| = |A||A^T| = |A|^2 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1.$

$|AA^T| = |I| = 1$

Thm A $n\times n$ (real matrix), then

1. A is orthogonal iff the rows are mutually orthogonal unit vectors in $\mathbb{R}^n$
2. " " cols " " " .

Proof $AA^T = I \Rightarrow \quad a_i \cdot a_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ ∎.

Fact if is A is 2×2 then $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

det $+1$ det -1.

recall if A is a symmetric matrix with n distinct eigenvalues, then it has orthogonal eigenvectors.

Thm An $n\times n$ real symmetric matrix with distinct eigenvalues can be diagonalized by an orthogonal matrix.

§10.1 Linear systems of differential equations

$$x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + g_1(t)$$

$$x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + g_2(t)$$

in general
$$x_1'(t) = a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + g_1(t)$$
$$\vdots$$
$$x_n'(t) = a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + g_n(t)$$

notation $X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ $\quad A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}$ $\quad G(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$

so we have $X'(t) = A(t)X(t) + G(t)$ or $X' = AX + G$.

note:
$A'(t) = \begin{bmatrix} a_{11}'(t) & \cdots & a_{1n}'(t) \\ \vdots & & \\ a_{n1}'(t) & \cdots & a_{nn}'(t) \end{bmatrix}$ gives product rule: $(AB)' = A'B + AB'$.
in particular if A constant matrix $(AX)' = AX'$

Def^n a linear system is homogeneous if $G(t) = \underline{0}$.

Special case $A =$ constant matrix (i.e. all functions are constant).

Example ① D diagonal $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ $X' = DX$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightsquigarrow \left.\begin{aligned} x_1' &= 2x_1 \\ x_2' &= -x_1 \end{aligned}\right\} \quad \begin{aligned} x_1 &= c_1e^{2t} \\ x_2 &= c_2e^{-t} \end{aligned} \qquad X = \begin{bmatrix} c_1e^{2t} \\ c_2e^{-t} \end{bmatrix}.$$

$$= c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}e^{-t}$$

② spose $X' = AX$ but $A = TDT^{-1}$, D diagonal.
T change of basis matrix (constant)

spose $S(t)$ solves $X' = DX$, i.e. $S' = DX'$ $\left(S = \begin{bmatrix} c_1e^{\lambda_1} \\ \vdots \\ c_ne^{\lambda_n} \end{bmatrix} \right)$.

claim TS solves $X' = AX$

check: $(TS)' = TS'$. $\quad ATS = TDT^{-1}TS = TDS = TS'$ ✓.