Example $\begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$ $\qquad \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$

what does A do to a combination of eigenvectors?

$$A(\alpha \underline{v}_1 + \beta \underline{v}_2) = A\alpha \underline{v}_1 + A\beta v_2 = \alpha\lambda_1 v_1 + \beta\lambda_2 v_2$$

Q: what is A in basis $\{\underline{v}_1, \underline{v}_2\}$? $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \uparrow T & & \uparrow T \\ \mathbb{R}^2 & \xrightarrow[D]{} & \mathbb{R}^2 \end{array}$$

D $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

T change of basis map

$$\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$$

$\{\underline{v}_1, \underline{v}_2\}$ $\qquad \{\underline{e}_1, \underline{e}_2\}$

$$\begin{bmatrix} a \\ b \end{bmatrix} \longmapsto a\underline{v}_1 + b\underline{v}_2 .$$

so $\quad A = T D T^{-1}$ or $D = T^{-1} A T \qquad T = [\underline{v}_1, \underline{v}_2]$.

Warning there is not always a basis of eigenvectors.

Example $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ eigenvalues $0, 0$. $\quad A - 0I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ just 1 eigenvector. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thm A set of eigenvectors with distinct eigenvalues are linearly independent.

Proof suppose $\lambda_1, \dots, \lambda_k$ with $A\underline{v}_i = \lambda_i \underline{v}_i$

~~suppose dependent~~ induction on number of vectors.

$n=1$ 1 non-zero vector always linearly independent.

n: suppose ① $c_1 v_1 + \dots + c_k \underline{v}_k = 0$ ~~for fewest~~ can assume all $c_i \neq 0$ as otherwise in smaller case

then ② $c_1 \lambda_1 v_1 + \dots + c_k \lambda_k \underline{v}_k = 0$

② $- \lambda_k$①: $\quad \underset{\neq 0}{c_1} \underset{\neq 0}{(\lambda_1 - \lambda_k)} \underline{v}_1 + \dots + \underset{\neq 0}{c_{k-1}} \underset{\neq 0}{(\lambda_{k-1} - \lambda_k)} \underline{v}_k = 0$ # works for $n-1$ □.

special case: symmetric matrices.

<u>Thm</u> The eigenvalues of a symmetric matrix are real.

<u>recall</u> complex conjugate $z = a+bi$ then $\bar{z} = a-bi$

can take complex conjugate of vectors/matrices. $\overline{\begin{bmatrix} i \\ -i \end{bmatrix}} = \begin{bmatrix} -i \\ i \end{bmatrix}$. etc...

$z \in \mathbb{C}$ is real iff $z = \bar{z}$

<u>Proof</u> spose $\underline{v}$ has eigenvalue $\lambda \in \mathbb{C}$, so $A\underline{v} = \lambda \underline{v}$

then $\bar{\underline{v}}^T A \underline{v} = \lambda \underbrace{\bar{\underline{v}}^T \underline{v}}$ $= \cancel{\|\underline{v}\|} \sum_{i=1}^{n} |v_i|^2 \in \mathbb{R}$.

so want to show LHS real:

$\overline{\bar{\underline{v}}^T A \underline{v}}^T = (\underline{v}^T \bar{A} \bar{\underline{v}})^T = \bar{\underline{v}}^T \bar{A}^T \underline{v} = \bar{\underline{v}}^T A \underline{v}$ as required. □.

$\bar{A} = A$, $\bar{A}^T = A$ symmetric.

<u>Thm</u> The eigenvectors of a symmetric matrix are orthogonal.

<u>Proof</u> let $A\underline{v}_1 = \lambda_1 \underline{v}_1$, $A\underline{v}_2 = \lambda_2 \underline{v}_2$ consider $\underline{v}_1^T A \underline{v}_2 = \lambda_2 \underline{v}_1^T \underline{v}_2 = \lambda_2 \underline{v}_1 \cdot \underline{v}_2$

$\underline{v}_1^T A \underline{v}_2 = \underline{v}_1^T A^T \underline{v}_2 = (A\underline{v}_1)^T \underline{v}_2 = \lambda_1 \underline{v}_1 \cdot \underline{v}_2$.

so $(\lambda_1 - \lambda_2)\, \underline{v}_1 \cdot \underline{v}_2 = 0$

$\Rightarrow \underline{v}_1 \cdot \underline{v}_2 = 0$ □.

<u>§9.2 Diagonalization</u>

<u>Defn</u> A matrix A is <u>diagonalizable</u> if there is a matrix P s.t. $P^{-1}AP = D$ diagonal.

<u>Thm</u> A is diagonalizable iff A has a basis of eigenvectors.

<u>Proof</u> $\Leftarrow$ change to basis of eigenvectors.

$\Rightarrow$ P $\Leftarrow$ columns of P are a basis of eigenvectors. □.

<u>Warning</u> don't always have a basis of eigenvectors.

Fact (Thm Jordan normal form).

give $A : \mathbb{R}^n \to \mathbb{R}^n$ $(n \times n)$ there is a basis $\underline{v}_1, \dots, \underline{v}_n$ s.t. TAT^{-1} is in block diagonal form, where each block is $\begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$

e.g.

$$\left[\begin{array}{c|cc|c|ccc} 1 & & & & & & \\ \hline & 2 & 1 & & & & \\ & & 2 & & & & \\ \hline & & & 2 & & & \\ \hline & & & & 3 & 1 & \\ & & & & & 3 & 1 \\ & & & & & & 3 \end{array}\right]$$

consequences / (useful facts about matrices).

① $\det(A) =$ product of eigenvalues (with multiplicity).

② $\operatorname{trace}(A) =$ sum of diagonal elements $=$ sum of eigenvalues (with multiplicity)

often we can find J without needing to find T. $(A = TJT^{-1})$

and can use ①,② to check our answer.

Example $A = \begin{bmatrix} 3 & 2 & 1 \\ -3 & -2 & -2 \\ 2 & 2 & 3 \end{bmatrix}$ find eigenvalues: $\det(A - \lambda I) = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} \quad \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & -\lambda & -1 \\ 0 & 2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(-3\lambda + \lambda^2 - 2) = (1-\lambda)(\lambda-2)(\lambda-1)$$

$$= -(\lambda-1)^2(\lambda-2) \qquad 1,1,2. \quad \text{so } TAT^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

to tell which one, check $\operatorname{rank}(A - I)$.

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ rank } 2 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$