algebraic definition: $\underset{2\times 2}{} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3×3: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

$= -a_{21} \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} + a_{22} \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix} - a_{23} \begin{vmatrix} \cdot & \cdot \\ \cdot & \cdot \end{vmatrix}$

4×4 $\begin{vmatrix} a & b & c & d \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = a \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} - b \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} + c \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} - d \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$.

Permutations a permutation is a bijection from $\{1,2,\dots n\}$ to itself.

eg $\overset{n=2}{\{1,2\}} \to \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$ $\quad$ $n=3$. $\begin{matrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{matrix}$ $\quad \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{matrix}$ etc..

$(2) = 2!$

Q: how many? A: $n!$

notation: $p: \{1,2,\dots n\} \to \{1,2,\dots n\}$
$i \mapsto p(i)$

Swaps: $\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{matrix}$ $\quad$ **fact** every permutation is a ~~product~~ composition of swaps.

"proof" $\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{matrix}$ $\quad \begin{matrix} 1 & 2 & 3 & 4 & \\ 4 & 2 & 3 & 1 & (14). \\ 4 & 1 & 3 & 2 & (\cancel{23}).(24). \\ 4 & 1 & 2 & 3. & (34). \end{matrix}$ $\square$.

Def^n $\sigma(p) = \begin{matrix} +1 \\ -1 \end{matrix}$ if p is a composition of an even/odd number of permutations

Fact this is well defined $\square$.

<u>Defn</u> $$\det(A) = \sum_p \sigma(p)\, a_{1p(1)} a_{2p(2)} \cdots a_{np(n)}$$

notation $\det(A) = |A|$.

<u>useful facts</u>

$|A^T| = |A|$

if A has a zero row or column, then $|A| = 0$ → ~~if A has the same~~

<u>Q</u>: how do row column operations change $|A|$?

- if B is formed from A by swapping two rows/cols, then $|B| = -|A|$.
- if B is formed from A by multiplying a row/col by $\alpha \neq 0$ then $|B| = \alpha|A|$.
- if B is formed from A by adding a multiple of one row/col to another, then $|B| = |A|$.

<u>note</u> if A has two rows (cols) the same, then $|A| = 0$. why?

A is non-singular iff $\det(A) \neq 0$.

if A, B are $n \times n$ then $|AB| = |A||B|$.

<u>note</u> we can use this to check how row operations change det.

- swaps. $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$. $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$ a single swap has $\sigma(p) = -1$! only one p has $a_{1p(1)} \cdots a_{np(n)} = 1$!
- row mult: 

α $\begin{vmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \alpha$.
- adding a multiple of a row $\begin{vmatrix} 1 & \alpha \\ 0 & 1 \end{vmatrix} = 1$

<u>Evaluation of determinants</u>

- using permutations is $O(n!)$
- using row operations is $O(n^3)$.

§8.5 Cramer's rule

$A\underline{x} = \underline{b}$ suppose A is non-singular, then $\underline{x} = A^{-1}\underline{b}$

Cramer's rule : $x_k = \frac{1}{|A|} \left| A(k, \underline{b}) \right|$

↑ matrix with k-th column replaced by $\underline{b}$

point : $\underline{x}$ depends continuously on $A, \underline{b}$.

Warning. consider

$$\left.\begin{aligned} u+v &= 2 \\ u+(1.001)v &= 2 \end{aligned}\right\} \text{ solutions } \begin{aligned} u&=2 \\ v&=0. \end{aligned}$$

$$\left.\begin{aligned} u+v &= 2 \\ u+(1.001)v &= 2.001 \end{aligned}\right\} \text{ solutions } \begin{aligned} u&=1 \\ v&=1. \end{aligned}$$

$A\underline{x} = \underline{b}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1.001 \end{bmatrix}$ $\quad \det(A) = 0.001$. $\quad \frac{1}{\det(A)} = 1000$.

row operations : $\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 1.001 & b \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 0.001 & b-a \end{array}\right]$ $\times 1000$

makes errors 1000 bigger....

§9.1 Eigenvalues and eigenvectors

Def$^{\underline{n}}$ suppose there is a number λ and a vector $\underline{v}$, $\underline{v} \neq 0$ such that $A\underline{v} = \lambda \underline{v}$

then ~~λ is an eigenvalue for A with eigenvect~~

$\underline{v}$ is an eigenvector for A with corresponding eigenvalue λ.

Example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A\underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \lambda = 1, \underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad A\underline{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \lambda = 0, \underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note if $\underline{v}$ is an eigenvector, so is $\alpha \underline{v}$ for any $\alpha \neq 0$.

most vectors not eigenvectors, e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

<u>Example</u> $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ eigenvector with eigenvalue 2.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ eigenvectors with eigenvalue 1

as is $\alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

<u>Q:</u> how do we find eigenvalues/eigenvectors?

consider $A\underline{v} = \lambda \underline{v}$

$A\underline{v} - \lambda I \underline{v} = 0$

$(A - \lambda I)\underline{v} = 0$ ($\underline{v} \neq 0$!) so $A - \lambda I$ is <u>singular</u>.

$\Rightarrow \det(A - \lambda I) = 0 \leftarrow$ degree n polynomial in λ.

solve for $\lambda_1, \dots, \lambda_n$ (may have repeats), then solve $A\underline{v} = \lambda_i \underline{v}$.

or $(A - \lambda_i I)\underline{v} = 0$.

<u>Example</u> $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 1$

$= \lambda^2 - 3\lambda + 2 - 1$

$= \lambda^2 - 3\lambda + 1$.

$\lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$.

to find eigenvalues: $\begin{bmatrix} 2 - \frac{3+\sqrt{5}}{2} & 1 \\ 1 & 1 - \frac{3+\sqrt{5}}{2} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix}$

gives $\underline{v} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$

$\boxed{\text{Example } \begin{bmatrix} 5 & 3 \\ 6 & -4 \end{bmatrix} \quad \lambda = 2, 1 \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.}$

<u>Defn</u> $\det(A - \lambda I)$ is a degree n polynomial in λ, called the characteristic polynomial for A.

<u>Summary</u> if $A\underline{v} = \lambda \underline{v}$ λ eigenvalue $\leftarrow$ solutions of $\det(A - \lambda I) = 0$

$\underline{v}$ eigenvector $\lambda_1, \dots, \lambda_n$ (with repeats).