Example $\begin{bmatrix} 1 & 1 & 1 & -1 & 5 \\ -1 & -1 & 0 & 0 & -3 \\ 2 & 2 & 2 & -2 & 11 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

§7.6 Nonhomogeneous systems

$\underset{n\times m}{A}\underline{x} = \underline{b}\ (\neq 0)$ $\qquad A : \mathbb{R}^m \to \mathbb{R}^n$.

Warning there may be <u>no</u> solutions

we call this <u>inconsistent</u>

$A : \mathbb{R}^m \to \mathbb{R}^n$

Example $2x_1 - 3x_2 = 6$

$4x_1 - 6x_2 = 8$

Observation spose $\underline{x}_1$ and $\underline{x}_2$ are solutions to $A\underline{x} = \underline{b}$ c!

then $A(\underline{x}_1 - \underline{x}_2) = A\underline{x}_1 - A\underline{x}_2 = \underline{b} - \underline{b} = \underline{0}$

how to solve $A\underline{x} = \underline{b}$

1. solve homogeneous problem $A\underline{x} = \underline{0}$ get solution set $K \subset \mathbb{R}^m$.
2. find any particular solution $A\underline{p} = \underline{b}$
3. general solution is $K + \underline{p}$.

2: row reduce augmented matrix $[A \mid \underline{b}]$.

if you get $0\ 0 \cdots 0 \mid 1$ then inconsistent.

otherwise there is a solution.

Example $\left[\begin{array}{ccc|c} -3 & 2 & 2 & 8 \\ 1 & 4 & -6 & 1 \\ 0 & -2 & 2 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$.

§7.7 Matrix inverses

Let A be $n\times n$ matrix. $\underset{n\times n}{B}$ is an <u>inverse</u> for A if $AB = BA = I_n$.

<u>notation</u> we often write A^{-1} for the inverse. $\mathbb{R}^n \underset{A^{-1}}{\overset{A}{\rightleftarrows}} \mathbb{R}^n$

<u>Warning</u> not all matrices have inverses. eg. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

<u>Def^n</u> if A has an inverse we call it non-singular
no ... singular

<u>useful facts about inverses</u>

1. $I_n^{-1} = I_n$
2. $(AB)^{-1} = B^{-1}A^{-1}$ check! $ABB^{-1}A^{-1} = AA^{-1} = I_n$
   $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
3. $(A^{-1})^{-1} = A$
4. $(A^t)^{-1} = (A^{-1})^t$
5. A^{-1} exists iff row echelon form of $A = I_n$.
6. A^{-1} exists iff $\operatorname{rank}(A) = n$.
7. AB non-singular $\Rightarrow A, B$ both non-singular.
9. all elementary matrices are non-singular.
10. A is non-singular iff $A\underline{x} = \underline{b}$ has a <u>unique</u> solution for each $\underline{b}$.

<u>Thm</u> $A\underline{x} = \underline{0}$ has a ~~unique~~ non-trivial solution iff A is singular

$A\underline{x} = \underline{b}$ has a unique solution iff A is non-singular, then $\underline{x} = A^{-1}\underline{b}$.

<u>Q</u>: how do we find the inverse?

2x2 case $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

general case: want to solve $AB = I_n$ $\quad I_n = \begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ 0 & 0 & \cdots \\ \vdots & \vdots & \\ 0 & 0 & \end{bmatrix}$.

can solve $A\underline{x_1} = \underline{e_1}$, $A\underline{x_2} = \underline{e_2}$, ... $A\underline{x_n} = \underline{e_n}$

then $A\left[\underline{x_1} \cdots \underline{x_n}\right] = \left[\underline{e_1} \cdots \underline{e_n}\right] = I_n$

so row reduce augmented matrix $[A|I]$.

<u>Example</u> ① $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{cb}{a} & -\frac{c}{a} & 1 \end{array}\right]$ $r2 - \frac{c}{a} r1$

$r1 - \dfrac{b}{\left(d - \frac{bc}{a}\right)} r2 = r1 - \dfrac{ab}{ad-bc} r1$ $\quad \cancel{\left[\begin{array}{cc|cc} a & 0 & 1 - \frac{(ad-bc)c}{ab\,a} & -\frac{ad-bc}{ab} \\ 0 & \frac{da-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]}$

$$\left[\begin{array}{cc|cc} a & 0 & 1 + \frac{ab}{ad-bc}\frac{c}{a} & -\frac{ab}{ad-bc} \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \quad \begin{array}{l} r1\,(ad-bc)/a. \\ a\,r2 \end{array}$$

$$\left[\begin{array}{cc|cc} ad-bc & 0 & \left(\frac{ad-bc+bc}{ad-bc}\right)\frac{ad-bc}{a} & -b \\ 0 & ad-bc & -c & a \end{array}\right] = \left[\begin{array}{cc|cc} ad-bc & 0 & d & -b \\ 0 & ad-bc & -c & a \end{array}\right].$$

② find A^{-1} when $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 1 \\ 1 & -2 & 0 \end{bmatrix}$

<u>Observation</u> · finding A^{-1} takes $O(n^3)$ operations for an $n \times n$ matrix.

· if A^{-1} does not exist, get a full row of zeros....

§8.1 Determinants

<u>Defn</u> (geometric) $A: \mathbb{R}^n \to \mathbb{R}^n$ $\quad \det(A) = \pm$ change in volume

$\det(A) = +$ volume of A(unit cube) if A preserves orientation

$\qquad\quad -$ volume of A(unit cube) if A reverses orientation.