$$= \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}.$$

Fact $\det(A_F) = \det(A_E)$.

Q: given a matrix $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ how do we find "best" basis?

§7.2 Row operations

matrix: rectangle of numbers. $\leftrightarrow$ linear map $\mathbb{R}^m \to \mathbb{R}^n$

$\leftrightarrow$ system of equations

$$3x + 2y = 1$$
$$x - y = 2$$

$$\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad A\underline{x} = \underline{b}$$

1. swap two rows of A
2. multiply a row of A by a non-zero number
3. add a scalar multiple of one row of A to another row of A.

Examples

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & -1 & 2 \end{array}\right] \underset{\text{swap } ①,②}{\rightsquigarrow} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 3 & 2 & 1 \end{array}\right] \underset{\text{mult ① by 3}}{\rightsquigarrow} \left[\begin{array}{cc|c} 3 & -3 & 2 \\ 3 & 2 & 1 \end{array}\right]$$

subtract r1 from r2

$$\rightsquigarrow \left[\begin{array}{cc|c} 3 & -3 & 2 \\ 0 & 5 & -1 \end{array}\right] \qquad y = \frac{1}{5} \qquad x = \frac{2+\frac{3}{5}}{3}.$$

Def^n An elementary matrix is one formed by applying a single row operation to I_n. e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{r1 \leftrightarrow r3}{\rightsquigarrow} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fact applying a row operation to A is the same as multiplying on the left by an appropriate E.

eg

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & -1 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 3 & 2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & -1 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & -3 & 2 \\ 0 & 5 & -1 \end{array}\right]$$

solve $A\underline{x} = \underline{b}$ $\qquad A : \mathbb{R}^m \to \mathbb{R}^n$

row operation: $\quad EA$ corresponds to changing basis in $\mathbb{R}^n$ (range)

doesn't change solution in $\mathbb{R}^m$!

§7.3.

Row echelon form

The leading entry of a row is the first non-zero term.

~~if any row has a leading term in column j, then all entries below that are zero~~. The column of the leading terms is strictly increasing as you go down the columns.

$$\begin{bmatrix} \boxed{2} & 2 & 3 & 4 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & \boxed{4} \end{bmatrix} \qquad \square \text{ leading terms/pivots}$$

Reduced row echelon form

- Leading entries are all equal to 1
- Leading term is the only non-zero entry in that column.

$$\begin{bmatrix} \boxed{1} & 1 & 3/2 & 2 \\ 0 & 0 & \boxed{1} & 8 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 1 & 1/2 & -1/2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Thm: Every matrix may be put in (reduced) row echelon form by row operations. The reduced row echelon form is unique. $\square$

§7.4 Row and column spaces

A $\quad n \times m$ matrix $\qquad A : \mathbb{R}^m \to \mathbb{R}^n$
rows × cols.

Defn: The row space is the span of the rows of A. $\leq \mathbb{R}^m$

column space " " " cols of A $\leq \mathbb{R}^n$

The row rank is the dimension of the row space $(\leq m)$

column rank $\qquad$ column space. $(\leq n)$

Thm: row rank = column rank.

Proof (sketch) show row operations don't change the rank, and rank = #pivots / leading terms □.

Recall can solve $\underline{A}\underline{x} = \underline{b}$ by row operations. $A: \mathbb{R}^{n} \to \mathbb{R}^{m}$
$m \times n$

① swap two rows of A. can do this by

Thm Let R be the (reduced) row echelon form of A. then there is a matrix E s.t. $R = \underset{m\times m}{E}\underset{m\times n}{A}$.

Proof $E = E_1 E_2 \dots E_r$ where E_i corresponds to the i-th row operation. □.

---

rank (R) = # non-zero rows of R.

Thm row operations do not change the row space.

Warning: row operations change the column space!

dimension of col space = dimension of the rows

Corollary if A is n×n then rank(A) = n iff reduced row echelon form of A = I_n.

§7.5 Homogeneous systems

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m = 0$$

$A: \mathbb{R}^m \to \mathbb{R}^n$

$A\underline{x} = \underline{0}$

(note $\underline{x} = \underline{0}$ always a solution)

use Gaussian elimination / row reduction.

dimension of solution set = # free variables = # cols without a leading entry.

check: solution set is a vector subspace of $\mathbb{R}^m$ called the kernel or null space of A.

$$\boxed{\text{dimension of solution set} = m - \operatorname{rank}(A).}$$

note $m > n \Rightarrow$ always non-trivial solns!