Warning $AB \neq BA$ in general $\qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$AB = 0 \not\Rightarrow A$ or $B = 0 \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Special matrices $\quad O$ zero matrix $(a \times b)$.

I identity matrix $(n \times n)$ $\quad \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Transpose $\quad [A^t]_{ij} = [A]_{ji} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Def^n A linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ satisfies $L(\underline{a} + \underline{b}) = L(\underline{a}) + L(\underline{b})$

$L(c\underline{a}) = cL(\underline{a})$.

Fact A linear map may be written as a matrix

choose a basis $\{e_1, \dots, e_m\}$ for $\mathbb{R}^m$

$\{f_1, \dots, f_n\}$ for $\mathbb{R}^n$

Composition

$$\mathbb{R}^m \xrightarrow{S} \mathbb{R}^n \xrightarrow{T} \mathbb{R}^p$$

$$\underline{x} \longmapsto S\underline{x} \longmapsto TS\underline{x}$$

then
$$L(e_1) = a_{11}f_1 + a_{21}f_2 + \dots + a_{n1}f_n$$
$$L(e_2) = a_{12}f_1 + a_{22}f_2 + \dots + a_{n2}f_n$$
$$L(e_m) = a_{1m}f_1 + a_{2m}f_2 + \dots + a_{nm}f_n$$

$\underline{v} \in \mathbb{R}^m$, then $\underline{v} = v_1e_1 + v_2e_2 + \dots + v_me_m \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$ in basis $\{e_1, \dots, e_m\}$.

$$L(\underline{v}) = v_1L(e_1) + v_2L(e_2) + \dots + v_mL(e_m)$$
$$= v_1(a_{11}f_1 + \dots + a_{n1}f_n) + \dots + v_m(a_{1m}f_1 + \dots + a_{nm}f_n)$$
$$= (a_{11}v_1 + a_{12}v_2 + \dots + a_{1m}v_m)f_1 + \dots + (a_{n1}v_1 + a_{n2}v_2 + \dots + a_{nm}v_m)f_n$$

$$L(v) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nm} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \text{ in basis } \{f_1, \dots, f_n\}.$$

Note $\quad e_1 \longmapsto \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}$ first col.

$e_i \longmapsto \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix}$ ith col

Geometric action of linear maps

Examples $A: \mathbb{R}^2 \to \mathbb{R}^2$ $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\begin{bmatrix}2&0\\0&1\end{bmatrix}$ $\begin{bmatrix}2&0\\0&-1\end{bmatrix}$ $\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $\begin{bmatrix}0&0\\0&0\end{bmatrix}$

Which transformations are linear? reflection in x-axis
translation by $(1,0)$.
rotation by θ degrees.
projection to y-axis.

Determinants for 2x2 matrices

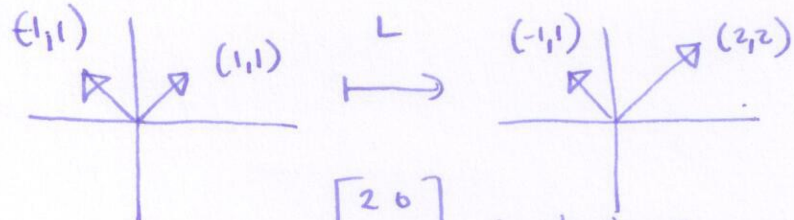
$\begin{vmatrix}a&b\\c&d\end{vmatrix} = ad-bc$ geometric interpretation: change of volume.

Inverses for 2x2 matrices

$A^{-1} = \dfrac{1}{\det(A)}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$ as long as $\det(A) \neq 0$.

Change of basis is a linear map

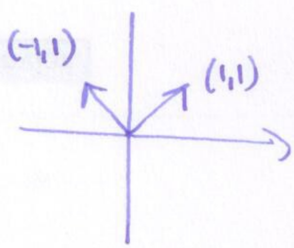
Example.



$\begin{bmatrix}2&0\\0&1\end{bmatrix}$ in basis $\{(1,1),(-1,1)\}$

$\{(1,1),(-1,1)\}$

Q: what is L wrt to the standard basis $\{e_1,e_2\}$?



$$\underline{v} = \underbrace{\begin{bmatrix}v_1\\v_2\end{bmatrix}}_{\{f_1,f_2\}} = v_1(1,1)+v_2(-1,1) = (v_1-v_2,\ v_1+v_2) = \underbrace{\begin{bmatrix}v_1-v_2\\v_1+v_2\end{bmatrix}}_{\{e_1,e_2\}}$$

$$\begin{bmatrix}1\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\1\end{bmatrix} \qquad \begin{bmatrix}0\\1\end{bmatrix} \mapsto \begin{bmatrix}-1\\1\end{bmatrix}$$

so change of basis matrix is $\begin{bmatrix}1&-1\\1&1\end{bmatrix}$

$$\begin{tikzcd} \mathbb{R}^2_F \arrow[r, "A_F"', "{\begin{bmatrix}2&0\\0&1\end{bmatrix}}"] \arrow[d, "T"'] & \mathbb{R}^2_F \arrow[d, "T"] \\ \mathbb{R}^2_E \arrow[r, "A_E"'] & \mathbb{R}^2_E \end{tikzcd}$$

so $A_E = T A_F T^{-1}$

$$= \begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&1\\-1&1\end{bmatrix}$$